5 Friday, September 1

Derivatives

Definition (Rates of Change)**.** The **average rate of change** of a function f is defined to be the slope of **secant lines**, or lines that meet a graph in at least two points.

$$\begin{aligned} Rate_{avg} &= \text{slope of secant line} \\ &= \frac{\text{change in } f}{\text{change in } x} \\ &= \frac{f(c+h)-f(c)}{h} \\ &= \text{"approximation of } f\text{"} \end{aligned}$$

The expression

$$\frac{f(c+h)-f(c)}{h}$$

is called a **difference quotient**.

When it comes to measuring how variables change with respect to one another, the average rate of change does have a flaw, namely that it is an average. Knowing the rate of change at a moment requires looking at **tangent lines**, or lines that meet a graph at a point, but do not cut the graph. Tangent lines are the result of a limiting process where the two points of a secant line come together.

Thus, to find the slope of a tangent line, take a limit as $h \to 0$:

$$\begin{aligned} f'(c) &= \text{slope of tangent line at } x = c \\ &= \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \end{aligned}$$

The limiting value is called the **derivative** of the function f at the point $x = c$. The slope of the tangent line is often called the **instantaneous rate of change**.

Note (Notation of derivatives)**.** There are many different ways to denote the derivative:

$$f'(x) \quad \frac{dy}{dx} \quad \frac{df}{dx} \quad y' \quad D_x[y] \quad \left.\frac{dy}{dx}\right|_{x=c}.$$

If $y = x^2 + x$, then

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^2 + x\right) = 2x + 1 \qquad \left.\frac{dy}{dx}\right|_{x=3} = 7.$$

Definition (Alternate definition of the derivative)**.** The derivative can also be expressed as the limit

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

Example (Finding derivatives of functions)**.** Using the definition of the derivative, find $f'(x)$.

(1) $f(x) = -3$

(2) $f(x) = 2x^2 - x + 3$

(3) $f(x) = \dfrac{3}{6-x}$

(4) $f(x) = \sqrt{2x+1}$

(5) $f(x) = \dfrac{1}{\sqrt{x}}$

Example (Finding derivatives and tangent lines)**.** What is the equation for the line tangent to the graph of f at the specified point?

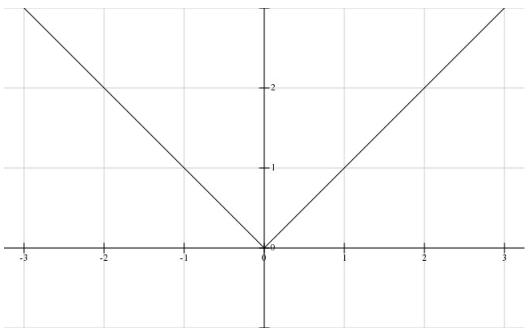
(1) $f(x) = x^3 - x$ at $c = -1$.

(2) $y = \dfrac{x-1}{x+1}$ at $c = 2$.

Note. The derivative can possibly not exist. There are two primary cases for when this happens.

(1) Corners

$$f(x) = |x|$$



(2) Vertical tangents

$$f(x) = \sqrt[3]{x}$$