ON A FORMULA ABOUT SCHUR FUNCTOR

1. Main result

The main propurse of this note is to prove the following formula about Schur functor in [2][Exercise 6.11]:

**Proposition 1.1.** Let $V$ and $W$ be vector spaces over $\mathbb{C}$ of dimension $n$ and $m$ respectively, let $\nu$ be a partition of length $d = n + m$, then there is a decomposition of $GL(V) \times GL(W)$ module:

\[(1.1) \quad S_\nu(V \oplus W) = \bigoplus N_{\mu\lambda\nu}(S_\mu V \otimes S_\lambda W),\]

where the sum to the right hand side is over all partitions $\mu, \lambda$ of length $a, b$ respectively with $a + b = d$, and $N_{\mu\lambda\nu}$ are the Littlewood-Richardson numbers.

Recall that the Schur functor $S_\nu$ can be defined as $S_\nu(V) = V \otimes^d \mathbb{C}(S_d) \cdot c_\nu$, here $c_\nu$ is the Young symmetrizer correspond to $\nu$ and the tensor product is over $\mathbb{C}(S_d)$. In [2], $\mathbb{C}(S_d) \cdot c_\nu$ is denoted by $V_\nu$, it is a irreducible representation of $S_d$ and all irreducible representations of $S_d$ are of this form[2][Theorem 4.3].

To prove this proposition, first we have a decomposition of vector spaces

\[(1.2) \quad (V \oplus W) \otimes^d = \bigoplus_{a+b=d} (V \otimes^a W \otimes^b) \otimes \mathbb{C}(S_a \times S_b).\]

Keep track of the action of $S_d$ on $(V \oplus W) \otimes^d$, we know the summands on the right hand side of 1.2 are stable under the action of $S_d$, and we can actually write them as $\bigotimes_{\sigma \in S_d/(S_a \times S_b)} (V \otimes^a W \otimes^b) \cdot \sigma$. But this is just $\text{Ind}_{S_a \times S_b}^{S_d} V \otimes^a W \otimes^b$.

So we have the following identity:

\[(1.3) \quad (V \oplus W) \otimes^d = \bigoplus_{a+b=d} ((V \otimes^a \otimes W \otimes^b) \otimes \mathbb{C}(S_a \times S_b) \mathbb{C}(S_d)),\]

tensoring $V_\nu$ on both side, we have:

\[(1.4) \quad S_\mu(V \oplus W) = \bigoplus_{a+b=d} (V \otimes^a \otimes W \otimes^b \otimes \mathbb{C}(S_a \times S_b) \text{Res}V_\nu),\]

here $\text{Res}$ is $\text{Res}_{S_a \times S_b}^{S_d}$.

We know all the irreducible representations of $S_a \times S_b$ are of the form $V_\mu \otimes V_\lambda$ where $\mu$ and $\lambda$ are partitions of $a$ and $b$ respectively[2][Exercise 2.36], and we have $(V \otimes^a \otimes W \otimes^b) \otimes (V_\mu \otimes V_\lambda) = S_\mu(V) \otimes S_\lambda(W)$. Let $M_{\mu\lambda\nu}$ be the multiplicities of $V_\mu \otimes V_\lambda$ in $\text{Res}V_\nu$, then to prove the proposition, one need to show:

**Lemma 1.2.** For all partitions $\nu, \mu$ and $\lambda$, the non-negative integers $M_{\mu\lambda\nu}$ and $N_{\mu\lambda\nu}$ are equal.

From Frobenius reciprocity, we have $M_{\mu\lambda\nu}$ is also the multiplicity of $V_\mu$ in $\text{Ind}_{S_a \times S_b}^{S_d} V_\lambda \otimes V_b$. 
2. The ring of representations of permutation groups

Let \( R_d = K_0(\mathbb{C}(S_d)) \) be the Grothendieck group of representations of \( S_d \), i.e., the free abelian group generated by isomorphic classes \([V]\) of representations of \( S_d \) modulo the relation \([V] + [W] = [U]\) if there is an exact sequence \( 0 \to V \to U \to W \to 0 \) of \( S_d \)-modules. Actually \( R_d \) is a free abelian group with basis \([V_\mu]\) for all partitions \( \mu \) of \( d \). Let \( R_0 = \mathbb{Z} \).

**Definition 2.1.** Let \( R = \bigotimes_{k \geq 0} R_k \) and define a function \( \circ : R_m \times R_n \to R_{m+n} \) by:

\[
(2.1) \quad [V] \circ [W] = \text{Ind}_{S_m \times S_n}^{S_{m+n}} V \otimes W
\]

Then from \( \text{Ind}_{S_m \times S_n}^{S_{m+n}} V \otimes W = \bigoplus_{g \in S_{m+n}/(S_m \times S_n)} (V \otimes W) \cdot g \) we can prove

\[
(2.2) \quad ([U] \circ [V]) \circ [W] = [U] \circ ([V] \circ [W]) = \text{Ind}_{S_{m+n}}^{S_{m+n}} U \otimes V \otimes W,
\]

i.e., \( R \) becomes a commutative graded ring under \( \circ \).

Let \( \Lambda \) be the inverse limit of rings of symmetric polynomials with coefficients in \( \mathbb{Z} \), and the inverse system is given by \( \varphi_{mn}(P(x_1, \ldots, x_m)) = P(x_1, \ldots, x_n, 0, \ldots, 0) \) when \( m > n \). We have \( \Lambda = \bigoplus \Lambda_n \) where \( \Lambda_n \) can be identified with the symmetric polynomials of degree \( d \) in \( k \geq d \) variables. Let \( H_\mu, S_\mu \) and \( M_\mu \) be the complete symmetric polynomials, Schur polynomials and monomial symmetric polynomials respectively, they form different basis of \( \Lambda \).

Let \( \mu = (\mu_1, \ldots, \mu_k) \) be a partition of \( d \), recall in [2], we define \( U_\mu = \text{Ind}_{S_{\mu_1} \times \ldots \times S_{\mu_k}}^{S_d} 1 \) be the representation of \( S_d \). Then we are going to prove the following main theorem:

**Theorem 2.2.** Define \( \phi : \Lambda \to R \) by \( \phi(H_\mu) = [U_\mu] \), then \( \phi \) is a well-defined isomorphism of graded rings with \( \phi(S_\mu) = [V_\mu] \).

Once we have the theorem, then by the Littlewood Richardson rule: \( S_\lambda \cdot S_\mu = \sum_{\nu} N_{\lambda \mu} S_\nu \), we have \([V_\lambda] \circ [V_\mu] = \bigoplus \nu N_{\lambda \mu} [V_\nu]\), which is Lemma 1.2.

3. The ring of symmetric polynomials and the proof of main theorem

Before we prove the main theorem, let us first go over some basic formulas about some important symmetric functions, in which lots of combinatoric constants appear.

**Complete symmetric functions** with \( m \) indeterminates are defined by the following identity of formal power series in \( t \):

\[
(3.1) \quad \prod_{i=1}^{m} \frac{1}{1 - x_i t} = \sum_{k=0}^{\infty} H_k(x_1, \ldots, x_m) t^k
\]

And if \( \mu = (\mu_1, \ldots, \mu_k) \), define \( H_\mu = H_{\mu_1} \cdots H_{\mu_k} \).

**Schur polynomials** are defined by

\[
(3.2) \quad S_\mu(x_1, \ldots, x_m) = \frac{|x_1^{\mu_1+m-i}|}{|x_1^{m-i}|}.
\]

\( S_\mu \) can be represented in terms of \( H_k \) by Jacobi-Trudi identity [2][A.5]:

\[
(3.3) \quad S_\mu = |H_{\mu_1+j-1}|.
\]
As a special case of the equation 3.3 when \( k = 1 \), i.e., \( \mu = (\mu_1) \), we have \( S(d) = H_d \). So
\[
H_\mu = S(\mu_1) \cdot \ldots \cdot S(\mu_k) = \sum_\nu K_{\nu\mu} S_\nu.
\]
The sum on the right runs over all partitions \( \nu \) of \( d = |\mu| \). \( K_{\nu\mu} \) are the Kostka numbers, if we apply Pieri’s rule to the second equation, we have the \( K_{\nu\mu} \) is the number of ways one can fill the boxes of the Young diagram of \( \nu \) with \( \mu_1 \) 1’s, \( \mu_2 \) 2’s, up to \( \mu_k \) k’s, in such a way that the entries in each row are nondecreasing, and those in each column are strictly increasing.

Especially, these integers \( K_{\nu\mu} \) satisfy \( K_{\mu\mu} = 1 \) and \( K_{\nu\mu} = 0 \) if \( \mu > \nu \), i.e., if the first nonvanishing \( \mu_i - \nu_i \) is positive. One can show this gives a total order on the set of partitions of \( d \), and if we arrange the \( S_\nu \) on the right hand side of 3.4 in the decreasing order, then the transition matrix \( (K_{\nu\mu}) \) is an upper triangle matrix with diagonal elements equal to 1. So it is invertible and its inverse is also a integer matrix, as a consequence of this, we have:

**Lemma 3.1.** All the \( U_\mu \) form a \( \mathbb{Z} \)-basis of \( R_d \) when \( \mu \) runs over all partitions of \( d \).

The assertion is from the following theorem[2][Corollary 4.39](which is proven in terms of the corresponded formula of characters) and the above discussions about \( K_{\nu\mu} \).

**Theorem 3.2** (Young’s rule). The integer \( K_{\nu\mu} \) is the multiplicity of the irreducible representation \( V_\nu \) in representation \( U_\mu \), i.e.:
\[
U_\mu = \bigoplus_\nu V_\nu^{\oplus K_{\nu\mu}}.
\]

Now we can prove Theorem 2.2:

**Proof.** From lemma 3.1, we have \( H_\mu \) form a basis of \( \Lambda \) and \( U_\mu \) form a basis of \( R \), so we know \( \phi \) is a well-defined surjective additive homomorphism. To show it is a ring isomorphism, it suffices to verify:
\[
U_\mu = U_{\mu_1} \circ \ldots \circ U_{\mu_k}.
\]
But this is just from the definition of \( U_\mu \) and 2.1. To prove the rest of the theorem, one compare the equation 3.4 and 3.5 and use the fact \( (K_{\nu\mu}) \) is invertible as an integer matrix.

**References**