ON A FORMULA ABOUT SCHUR FUNCTOR

1. Main result

The main propurse of this note is to prove the following formula about Schur functor in [2][Exercise 6.11]:

Proposition 1.1. Let V and W be vector spaces over $\mathbb{C}$ of dimension n and m respectively, let ν be a partition of length $d = n+m$, then there is a decomposition of $GL(V) \times GL(W)$ module:

$$\mathbb{S}_\nu(V \oplus W) = \bigoplus N_{\mu\lambda\nu}(\mathbb{S}_\mu V \otimes \mathbb{S}_\lambda W), \tag{1.1}$$

where the sum to the right hand side is over all partitions μ, λ of length a, b respectively with $a + b = d$, and $N_{\mu\lambda\nu}$ are the Littlewood-Richardson numbers.

Recall that the Schur functor $\mathbb{S}_\nu$ can be defined as $\mathbb{S}_\nu(V) = V^{\otimes d} \otimes \mathbb{C}(S_d) \cdot c_\nu$, here c_ν is the Young symmetrizer correspond to ν and the tensor product is over $\mathbb{C}(S_d)$. In [2], $\mathbb{C}(S_d) \cdot c_\nu$ is denoted by V_ν, it is a irreducible representation of S_d and all irreducible representations of S_d are of this form[2][Theorem 4.3].

To prove this proposition, first we have a decomposition of vector spaces

$$(V \oplus W)^{\otimes d} = \bigoplus_{a+b=d} (V^{\otimes a} \otimes W^{\otimes b})^{\oplus \frac{d!}{a!b!}}. \tag{1.2}$$

Keep track of the action of S_d on $(V \oplus W)^{\otimes d}$, we know the summands on the right hand side of 1.2 are stable under the action of S_d, and we can actually write them as $\oplus_{\sigma \in S_d/(S_a \times S_b)}(V^{\otimes a} \otimes W^{\otimes b}) \cdot \sigma$. But this is just $\mathrm{Ind}_{S_a \times S_b}^{S_d} V^{\otimes a} \otimes W^{\otimes b}$.
So we have the following identity:

$$(V \oplus W)^{\otimes d} = \bigoplus \big((V^{\otimes a} \otimes W^{\otimes b}) \otimes_{\mathbb{C}(S_a \times S_b)} \mathbb{C}(S_d)\big), \tag{1.3}$$

tensoring V_ν on both side, we have:

$$\mathbb{S}_\mu(V \oplus W) = \bigoplus \big(V^{\otimes a} \otimes W^{\otimes b} \otimes_{\mathbb{C}(S_a \times S_b)} \mathrm{Res} V_\nu\big), \tag{1.4}$$

here Res is $\mathrm{Res}_{S_a \times S_b}^{S_d}$.

We know all the irreducible representations of $S_a \times S_b$ are of the form $V_\mu \otimes V_\lambda$ where μ and λ are partitions of a and b respectively[2][Exercise 2.36], and we have $(V^{\otimes a} \otimes W^{\otimes b}) \otimes (V_\mu \otimes V_\lambda) = \mathbb{S}_\mu(V) \otimes \mathbb{S}_\lambda(W)$. Let $M_{\mu\lambda\nu}$ be the multiplicities of $V_\mu \otimes V_\lambda$ in $\mathrm{Res}V_\nu$, then to prove the proposition, one need to show:

Lemma 1.2. *For all partitions ν, μ and λ, the non-negative integers $M_{\mu\lambda\nu}$ and $N_{\mu\lambda\nu}$ are equal.*

From Frobenius reciprocity, we have $M_{\mu\lambda\nu}$ is also the multiplicity of V_ν in $\mathrm{Ind}_{S_a \times S_b}^{S_d} V_a \otimes V_b$.

2. The ring of representations of permutation groups

Let $R_d = K_0(\mathbb{C}(S_d))$ be the Grothendieck group of representations of S_d, i.e. the free abelian group generated by isomorphic classes $[V]$ of representations of S_d modulo the relation $[V]+[W]=[U]$ if there is an exact sequence $0 \to V \to U \to W \to 0$ of of S_d-modules. Actually R_d is a free abelian group with basis $[V_\mu]$ for all partitions μ of d. Let $R_0 = \mathbb{Z}$.

Definition 2.1. Let $R = \bigotimes_{k\geq 0} R_k$ and define a function $\circ : R_m \times R_n \to R_{m+n}$ by:

$$[V]\circ[W] = [\mathrm{Ind}_{S_m\times S_n}^{S_{m+n}} V\otimes W] \tag{2.1}$$

Then from $\mathrm{Ind}_{S_m\times S_n}^{S_{m+n}} V\otimes W = \bigoplus_{g\in S_{m+n}/(S_m\times S_n)}(V\otimes W)\cdot g$ we can prove

$$([U]\circ[V])\circ[W] = [U]\circ([V]\circ[W]) = [\mathrm{Ind}_{S_l\times S_m\times S_n}^{S_{l+m+n}} U\otimes V\otimes W], \tag{2.2}$$

i.e., R becomes a commutative graded ring under $\circ$.

Let Λ be the inverse limit of rings of symmetric polynomials with coefficients in $\mathbb{Z}$, and the inverse system is given by $\varphi_{mn}(P(x_1,\ldots,x_m)) = P(x_1,\ldots,x_n,0,\ldots,0)$ when $m>n$. We have $\Lambda = \oplus\Lambda_d$ where Λ_n can be can identified with the symmetric polynomials of degree d in $k\geq d$ variables. Let H_μ, S_μ and M_μ be the complete symmetric polynomials, Schur polynomials and monomial symmetric polynomials respectively, they form different basis of Λ.

Let $\mu = (\mu_1,\ldots,\mu_k)$ be a partition of d, recall in [2], we define $U_\mu = \mathrm{Ind}_{S_{\mu_1}\times\ldots\times S_{\mu_k}}^{S_d} \mathbb{1}$ be the representation of S_d. Then we are going to prove the following main theorem:

Theorem 2.2. *Define $\phi : \Lambda \to R$ by $\phi(H_\mu) = [U_\mu]$, then ϕ is a well-defined isomorphism of graded rings with $\phi(S_\mu) = [V_\mu]$.*

Once we have the theorem, then by the Littlewood Richardson rule: $S_\lambda \cdot S_\mu = \sum_\nu N_{\mu\lambda\nu} S_\nu$, we have $[V_\lambda]\circ[V_\mu] = \oplus_\nu N_{\mu\lambda\nu}[V_\nu]$, which is Lemma 1.2.

3. The ring of symmetric polynomials and the proof of main theorem

Before we prove the main theorem, let us first go over some basic formulas about some important symmetric functions, in which lots of combinatoric constants appear.

Complete symmetric functions with m indeterminates are defined by the following identity of formal power series in t:

$$\prod_{i=1}^{m} \frac{1}{1-x_i t} = \sum_{k=0}^{\infty} H_k(x_1,\ldots,x_m)t^k \tag{3.1}$$

And if $\mu = (\mu_1,\ldots,\mu_k)$, define $H_\mu = H_{\mu_1}\cdot\ldots\cdot H_{\mu_k}$.

Schur polynomials are define by

$$S_\mu(x_1,\ldots,x_m) = \frac{|x_j^{\mu_i+m-i}|}{|x_j^{m-i}|}. \tag{3.2}$$

S_μ can be represented in terms of H_k by Jacobi-Trudi identity [2][A.5]:

$$S_\mu = |H_{\mu_i+j-i}|. \tag{3.3}$$

As a special case of the equation 3.3 when $k = 1$, i.e., $\mu = (\mu_1)$, we have $S_{(d)} = H_d$. So

$$H_\mu = S_{(\mu_1)} \cdot \ldots \cdot S_{(\mu_k)} = \sum_\nu K_{\nu\mu} S_\nu. \tag{3.4}$$

The sum on the right runs over all partitions ν of $d = |\mu|$. $K_{\nu\mu}$ are the Kostka numbers, if we apply Pieri's rule to the second equation, we have the $K_{\nu\mu}$ is the number of ways one can fill the boxes of the Young diagram of ν with μ_1 1's, μ_2 2's, up to μ_k k's, in such a way that the entries in each row are nondecreasing, and those in each column are strictly increasing.
Especially, these integers $K_{\nu\mu}$ satisfy $K_{\mu\mu} = 1$ and $K_{\nu\mu} = 0$ if $\mu > \nu$, i.e., if the first nonvanishing $\mu_i - \nu_i$ is positive. One can show this gives a total order on the set of partitions of d, and if we arrange the S_ν on the right hand side of 3.4 in the decreasing order, then the transition matrix $(K_{\nu\mu})$ is an upper triangle matrix with diagonal elements equal to 1. So it is invertible and its inverse is also a integer matrix, as a consequence of this, we have:

Lemma 3.1. *All the U_μ form a $\mathbb{Z}$-basis of R_d when μ runs over all partitions of d.*

The assertion is from the following theorem[2][Corollary 4.39](which is proven in terms of the corresponded formula of characters) and the above discussions about $K_{\mu\nu}$.

Theorem 3.2 (Young's rule)**.** *The integer $K_{\nu\mu}$ is the multiplicity of the irreducible representation V_ν in representation U_μ, i.e.:*

$$U_\mu = \bigoplus_\nu V_\nu^{\oplus K_{\nu\mu}}. \tag{3.5}$$

Now we can prove Theorem 2.2:

Proof. From lemma 3.1, we have H_μ form a basis of Λ and U_μ form a basis of R, so we know ϕ is a well-defined surjective additive homomorphism. To show it is a ring isomorphism, it suffices to verify:

$$U_\mu = U_{\mu_1} \circ \cdots \circ U_{\mu_k}. \tag{3.6}$$

But this is just from the definition of U_μ and 2.1. To prove the rest of the theorem, one compare the equation 3.4 and 3.5 and use the fact $(K_{\nu\mu})$ is invertible as an integer matrix. □

References

1. William Fulton. Young Tableaux, with Applications to Representation Theory and Geometry. Cambridge University Press, 1997.
2. W. Fulton and J. Harris, Representation theory, a first course. Graduate Texts in Mathematics, vol. 129, Springer-Verlag, New York, 1991.