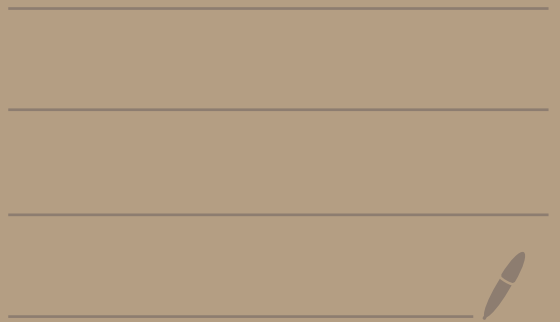


AG Notes 2024



1. Introduction

(1)

Roughly:

- An n -dim'l manifold is a space which looks locally like \mathbb{R}^n .
- A Riemann surface is a space which looks locally like \mathbb{C} .
- An algebraic variety over an alg. closed field k is a space which looks locally like an affine variety (Review later)

A few questions:

Q1 what does "locally look like" mean?

Q2 what's the difference between a 2-dim'l manifold and a Riem. surf?

One way to answer Q1 is purely topologically.

(2)

Def. An n -dim topological manifold is a (metrizable) topological space X s.t. $\forall x \in X$, \exists nbhd $U \in \mathcal{U} \xrightarrow[\text{homeo}]{\sim} \text{Ball} \subset \mathbb{R}^n$
(This is called a chart.)

Although precise, this misses the fact that on a manifold, one might want to talk about C^∞ functions, or on a Riem. surf. one might want holomorphic functions. Finally when working with alg. varieties the Zariski top is so coarse that a purely topological def is pretty much useless.

In order to make a better definition, we want to specify a distinguished collection of functions on open subsets of X which correspond to C^∞ /holomorphic/regular functions on the local models.

2. Sheaves

(3)

Let X be a topological space and k a field.

Def A presheaf of k -valued functions, \mathcal{F} , is an assignment to each open $U \subset X$,

$$\mathcal{F}(U) \subseteq \text{Maps}(U, k)$$

which is closed under restriction

$$\text{i.e. } f \in \mathcal{F}(U), \quad V \subset U, \Rightarrow f|_V \in \mathcal{F}(V)$$

Def A sheaf of k -valued functions \mathcal{F} is a presheaf s.t. $\forall U$ and open cover $\{U_i\}$ of U , $f \in \mathcal{F}(U) \Leftrightarrow \forall i, f|_{U_i} \in \mathcal{F}(U_i)$

The idea is the condition $f \in \mathcal{F}(U)$ can be checked locally.

Ex 1. Let $X = \mathbb{R}$, $\mathcal{F}(U) = \text{const. fun. } f: U \rightarrow \mathbb{R}$

Clearly, \mathcal{F} is a presheaf. Is it a sheaf?

Ex 2 Let X be arbitrary.

$$C(U) = \text{continuous functions } U \rightarrow \mathbb{R}.$$

This is a sheaf because continuity is local.

Ex 3 Let $X = \mathbb{R}^n$.

$$C^\infty(U) = \text{C}^\infty \text{ functions } U \rightarrow \mathbb{R}$$

This also a sheaf.

Ex 4 $X = \mathbb{C}$,

$$\mathcal{O}(U) = \text{holomorphic fns } U \rightarrow \mathbb{C}.$$

This is a sheaf.

3. Affine Varieties

Let $k = \text{alg}$ closed field.

$$A_k^n = k^n \text{ is affine space.}$$

$$S \subseteq k[x_1, \dots, x_n] = \mathbb{R}$$

$$V(S) := \{ a \in A_k^n \mid \forall f \in S, f(a) = 0 \}$$

Given $X \subseteq A_k^n$,

$$\mathcal{I}(X) = \{ f \in k[x_1, \dots, x_n] \mid \forall a \in X, f(a) = 0 \}$$

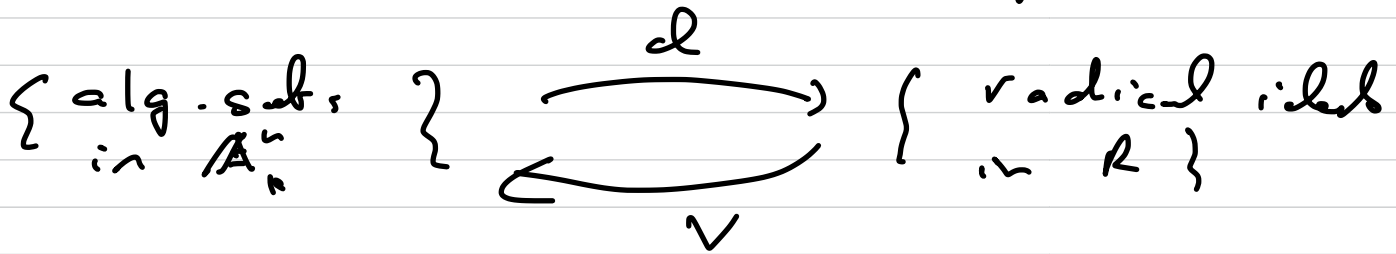
(5)

A set $X \subseteq \mathbb{A}^n$ is algebraic if $X = V(S)$, for some S .

An ideal $I \subset R$ is called radical if $f^n \in I \Rightarrow f \in I$

Thm (Hilbert; Nullstellensatz)

There are inverse bijections



Thm / Def

There is a topology on \mathbb{A}^n , called the Zariski topology, whose closed sets are precisely the algebraic sets. A basis for the open sets is given by

$$\begin{aligned} D(f) &= \mathbb{A}^n - V(f) \\ &= \{ a \mid f(a) \neq 0 \} \end{aligned}$$

Given a closed (= algebra) set X ,
 we give the induced topology
 X is called **irreducible** if it is
not the union of 2 proper closed
 subs.

Th H.15. Null. Cont

X is irred. $\Leftrightarrow \mathcal{O}(X)$ is prime
 So we have a bijection

$$\left\{ \begin{array}{l} \text{irred closed} \\ \text{sets in } \mathbb{A}_k^n \end{array} \right\} \begin{array}{c} \xrightarrow{\mathcal{O}} \\ \xleftarrow{\vee} \end{array} \left\{ \begin{array}{l} \text{prime ideals} \\ \text{of } R \end{array} \right\}$$

Given X closed, the **coordinate ring**

$$\mathcal{O}(X) = \frac{R}{\mathcal{I}(X)}$$

This is a finitely gen. reduced
 k -algebra. It is an integral domain
 if X is irred.

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Given $f \in R$, it defines a function called a **regular function**

$$\bar{f}: X \rightarrow k \text{ by } \bar{f}(a) = f(a)$$

Given $g \in \mathcal{O}(X)$, we have $\overline{f+g} = \bar{f}$ because $g = 0$ on X . Therefore \bar{f} can be identified with the image of $f \in \mathcal{O}(X)$.

Now suppose X is irred. Then $\mathcal{O}(X)$ is a domain, so it has a field of fractions, $k(X)$, called the **function field** of X .

Let us give $X \subset \mathbb{A}^n_k$ the induced topology. The basic opens are given by

$$\mathcal{D}(f) = \{ a \in X \mid f(a) \neq 0 \}$$

for $f \in \mathcal{O}(X)$.

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We embed $\mathcal{D}(f) \hookrightarrow \mathbb{A}^{n+1}$ by
 $(a_1, \dots, a_n) \mapsto (a_1, \dots, a_n, 1/f(a_1, \dots, a_n))$

This identifies the image with
 $\{(a_1, \dots, a_n, a_{n+1}) \mid (a_1, \dots, a_n) \in X, a_{n+1} f(a_1, \dots, a_n) = 1\}$

It's coordinate ring

$$\begin{aligned} \mathcal{O}(\mathcal{D}(f)) &\cong \mathcal{O}(X)[t] / (ft - 1) \\ &\cong \mathcal{O}(X)[\frac{1}{f}] \subset k(X) \end{aligned}$$

If $U = \bigcup_i \mathcal{D}(f_i)$ is open, define

$$\mathcal{O}_X(U) = \bigcup_i \mathcal{O}(\mathcal{D}(f_i)) \subset k(X)$$

Essentially by definition

Thm $U \mapsto \mathcal{O}_X(U)$ is a sheaf
 called the sheaf of regular functions or
 structure sheaf \mathcal{O}_X

4. Concrete Ringed Spaces

Fix a field k .

Def A concrete **ringed space**

— consists of a space X together with a sheaf \mathcal{F} of k -valued functions s.t.
 $\mathcal{F}(U) \subseteq \text{Map}(U, k)$
 is a sub k -algebra $\forall U$.

Many of the examples encountered earlier (\mathbb{R}^n, C^∞) , ..., an affine variety with \mathcal{O}_X are ringed spaces.

Def A **morphism**

$$f: (X, \mathcal{F}) \rightarrow (Y, \mathcal{G})$$

between concrete ringed spaces, is a continuous map $f: X \rightarrow Y$ s.t.
 $f^* \mathcal{G} = \mathcal{F} \circ f^{-1} \forall g \in \mathcal{G}(U)$

This is an **isomorphism** if f^{-1} is defined and is a morphism.

Let us say (X, \mathcal{F}) & (Y, \mathcal{G})
 are locally isomorphic if \exists open covers
 $\{U_i\}$ of X & $\{V_i\}$ of Y with
 $(U_i, \mathcal{F}) \cong (V_i, \mathcal{G})$

We can now define

Def A C^∞ manifold is a concrete
 ring space (X, C^∞) s.t. X is
 metrizable and locally isomorphic to
 (\mathbb{R}^n, C^∞) (then $k = \mathbb{R}$)

A Riemann surface is a ringed
 space (X, \mathcal{O}_X) locally isomorphic
 to \mathbb{C} with it's sheaf of holomorphic
 functions, ($k = \mathbb{C}$)

Def A prevariety in the sense of
 Serre is a ringed space (X, \mathcal{O}_X)
 s.t. every pt has an open nbhd
 isomorphic to an affine variety

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To be a variety, we need an analogue of the Hausdorff condition.

Serre's theory is described in his paper *Faisceaux Algébriques Cohérents* (= *Cohérent Algebraic Sheaves*) is a predecessor to scheme theory, which we'll discuss next.

Sheaves

It is convenient to define sheaves of things more general than functions. Here is the definition

Def Given a top space X , a **presheaf** of sets, groups, ... is

an assignment

$$X \supseteq U \xrightarrow{\text{open}} \mathcal{F}(U) \text{ a set, group, ...}$$

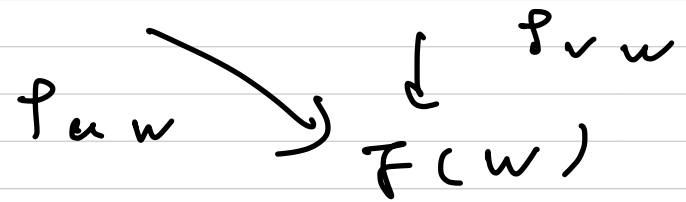
To each pair $U \supseteq V$, a map, homomorphism...

$$p_{UV} : \mathcal{F}(U) \rightarrow \mathcal{F}(V)$$

such that

$$p_{UU} = \text{id}$$

$$\text{and } \mathcal{F}(U) \xrightarrow{p_{UV}} \mathcal{F}(V)$$



commutes for any chain: $U \supseteq V \supseteq W$

A slicker way to formulate this is to make $\text{Open}(X)$ into a category. Then a presheaf is just a contravariant functor $\text{Open}(X) \rightarrow \text{Sets}$

In the previous examples
 \mathcal{F}_U is just restriction of functions.
We will use the same notation.

Def A presheaf \mathcal{F} is called a sheaf if $\forall U$ and open cover $\{U_i\}$
 Given $f_i \in \mathcal{F}(U_i)$ s.t.
 $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$
 \exists a unique $f \in \mathcal{F}(U)$ s.t.
 $f|_{U_i} = f_i$

All previous examples of sheaves are
sheaves in this sense. We
will treat more general examples
next.

6. Affine Schemes.

(14)

Thm / Def

The set $X = \text{Spec } R$ of prime ideals of R has a topology, also called the Zariski topology, with basic opens

$$D(f) = \{ \mathfrak{p} \in \text{Spec } R \mid f \notin \mathfrak{p} \}$$

This possesses a sheaf of commutative rings \mathcal{O}_x with

$$\mathcal{O}_x(D(f)) \cong R\left[\frac{1}{f}\right]$$

and $\rho_{D(f), D(gf)}$ the natural

$$\text{homomorphism } R\left[\frac{1}{f}\right] \rightarrow R\left[\frac{1}{fg}\right]$$

[See Hartshorne pp 70-72 for details]

The pair (X, \mathcal{O}_x) is called the **affine scheme** associated to R .

It is worth while comparing with the classical story. If $X \subset \mathbb{A}_k^n$ is algebraic with k alg. closed, let $R = \mathcal{O}(X)$.

Then Hilbert's Nullstellensatz we have a bijection between

$$X \leftrightarrow \text{Max ideals of } R \subsetneq \text{Spec } R.$$

The structure sheaf of $\text{Spec } R$ can be pulled back to X and it coincides with the sheaf of regular functions constructed earlier.

So scheme theory vastly extends the classical picture.