AG Notes 2024

1. Introduction

Roughly:

- An n dim'l manifold is a space which looks locally like $\mathbb{R}^n$.
- A Riemann surface is a space which looks locally like $\mathbb{C}$
- An algebraic variety over an alg. closed field k is a space which looks locally like an affine variety (Reviewed later)

A few questions:

Q1 what does "locally look like" mean?

Q2 what's the difference between a 2-dim'l manifold and a Riem. surface?

One way to answer Q1 is purely topologically.

Def An n-dim'l topological manifold is a (metrizable) topological space X s.t. $\forall x \in X$, $\exists$ nbhd $x \in U \xrightarrow[\text{homeo}]{\sim} \text{Ball} \subset \mathbb{R}^n$

(This is called a chart.)

Although precise, this misses the fact that on a manifold, one might want to talk about C^∞ functions, and on a Riem. surf. one might want holomorphic functions. Finally when working with alg. varieties the Zariski top is so coarse that a purely topological def is pretty much useless.

In order to make a better definition, we want to specify a distinguished collection of functions on open subsets of X which correspond to C^∞/holomorphic/regular functions on the local models.

2. Sheaves

Let X be a topological space and k a field.

Def A presheaf of k-valued functions $\mathcal{F}$ is an assignment to each open $U \subset X$,

$$\mathcal{F}(U) \subseteq \text{Maps}(U, k)$$

which is closed under restriction

i.e. $f \in \mathcal{F}(U)$, $V \subset U$, $\Rightarrow f|_V \in \mathcal{F}(V)$

Def A sheaf of k-valued fctn $\mathcal{F}$ is a presheaf s.t. $\forall\, U$ and open covers $\{U_i\}$ of U, $f \in \mathcal{F}(U) \Leftrightarrow$

$\forall\, i$, $f|_{U_i} \in \mathcal{F}(U_i)$

The idea is the condition $f \in \mathcal{F}(U)$ can be checked locally.

Ex 1. Let $X = \mathbb{R}$, $\mathcal{F}(U) =$ const. fnctn $f: U \to \mathbb{R}$

Clearly, $\mathcal{F}$ is a presheaf. Is it a sheaf?

Ex 2 Let X be arbitrary.

$C(U)$ = continuous functions from $U \to \mathbb{R}$.

This is a sheaf because continuity is local.

Ex 3 Let $X = \mathbb{R}^n$,

$C^\infty(U) = C^\infty$ functions $U \to \mathbb{R}$

This also a sheaf.

Ex 4 $X = \mathbb{C}$,

$\mathcal{O}(U) =$ holomorphic fns $U \to \mathbb{C}$.

This is a sheaf.

3. Affine Varieties

Let k = alg closed field.

$\mathbb{A}^n_k = k^n$ is affine space.

$S \subseteq k[x_1, \ldots x_n] = R$

$V(S) := \{a \in \mathbb{A}^n_k \mid \forall f \in S,\ f(a) = 0\}$

Given $X \subseteq \mathbb{A}^n_k$,

$I(X) = \{f \in R \mid \forall a \in X,\ f(a) = 0\}$

A set $X \subseteq \mathbb{A}^n$ is algebraic if $X = V(S)$, for some S.

An ideal $I \subset R$ is called radical if $f^n \in I \Rightarrow f \in I$

Thm (Hilbert's Nullstellensatz)

There are inverse bijections

$$\begin{Bmatrix} \text{alg. sets} \\ \text{in } \mathbb{A}^n_k \end{Bmatrix} \underset{V}{\overset{\mathcal{I}}{\rightleftarrows}} \begin{Bmatrix} \text{radical ideals} \\ \text{in } R \end{Bmatrix}$$

Thm/Def There is a topology on $\mathbb{A}^n_k$, called the Zariski topology, whose closed sets are precisely the algebraic sets. A basis for the open sets is given by

$$D(f) = \mathbb{A}^n - V(f)$$
$$= \{ a \mid f(a) \neq 0 \}$$

Given a closed (= algebraic) set X, we give the induced topology. X is called irreducible if it is not the union of 2 proper closed sets.

Thm Hilb. Null. Conf

X is irred. $\Leftrightarrow$ $\mathcal{I}(X)$ is prime

So we have a bijection

$$\left\{ \begin{array}{c} \text{irred closed} \\ \text{sets in } \mathbb{A}^n_k \end{array} \right\} \underset{V}{\overset{\mathcal{I}}{\rightleftarrows}} \left\{ \begin{array}{c} \text{prime ideals} \\ \text{of } R \end{array} \right\}$$

Given X closed, the coordinate ring

$$\mathcal{O}(X) = \frac{R}{\mathcal{I}(X)}.$$

This is a finitely gen. reduced k-algebra. It is an integral domain if X is irred.

Given $f \in R$, f defines a function called a regular function

$$\bar{f} : X \to k \text{ by } \bar{f}(a) = f(a)$$

Given $g \in \mathcal{I}(X)$, we have $\overline{f+g} = \bar{f}$ because $g = 0$ on X. Therefore $\bar{f}$ can be identified with the image of $f \in \mathcal{O}(X)$.

Now suppose X is irred. Then $\mathcal{O}(X)$ is a domain, so it has a field of fractions, $K(X)$, called the function field of X.

Let us give $X \subset \mathbb{A}^n_k$ the induced topology. The basic opens are given by

$$D(f) = \{ a \in X \mid f(a) \neq 0 \}$$

for $f \in \mathcal{O}(X)$.

We embed $D(f) \hookrightarrow \mathbb{A}^{n+1}$ by

$$(a_1, \dots a_n) \longmapsto (a_1, \dots a_n, 1/f(a_1, \dots a_n))$$

This identifies the image with

$$\{(a_1, \dots a_n, a_{n+1}) \mid (a_1 \dots a_n) \in X$$
$$a_{n+1} f(a_1 \dots a_n) = 1\}$$

It's coordinate ring

$$\mathcal{O}(D(f)) \cong \mathcal{O}(X)[t]/(ft-1)$$
$$\cong \mathcal{O}(X)[\tfrac{1}{f}] \subset k(X)$$

If $U = \bigcup_i D(f_i)$ is open, define

$$\mathcal{O}_X(U) = \bigcap_i \mathcal{O}(D(f_i)) \subset k(X)$$

Essentially by definition

Thm $U \longmapsto \mathcal{O}_X(U)$ is a sheaf

Called the sheaf of **regular functions** or **structure sheaf** $\mathcal{O}_X$

4. Concrete Ringed Spaces

Fix a field k.

Def A concrete ringed space consists of a space X together with a sheaf $\mathcal{F}$ of k-valued functions s.t. $\mathcal{F}(U) \subseteq \mathrm{Map}(U, k)$ is a sub k-algebra $\forall U$.

Many of the examples encountered earlier ($C(\mathbb{R}^n)$, C^∞), ..., an affine variety with $\mathcal{O}_X$ are ringed spaces.

Def A morphism

$$f : (X, \mathcal{F}) \to (Y, \mathcal{G})$$

between concrete ringed spaces, is a continuous map $f : X \to Y$ s.t.

$$f^* g = g \circ f \in \mathcal{F}(f^{-1} U) \ \forall g \in \mathcal{G}(U).$$

This is an isomorphism if f^{-1} is defined and is a morphism.

Let us say $(X, \mathcal{F})$ & $(Y, \mathcal{G})$ are locally isomorphic if $\exists$ open covers $\{U_i\}$ of X & $\{V_i\}$ of Y with

$$(U_i, \mathcal{F}) \cong (V_i, \mathcal{G})$$

We can now define

Def A C^∞ manifold is a concrete ring space (X, C_X^∞) s.t. X is metrizable and locally isomorphic to $(\mathbb{R}^n, C^\infty)$ (then $k = \mathbb{R}$)

A Riemann surface is a ringed space $(X, \mathcal{O}_X)$ locally isomorphic to $\mathbb{C}$ with its sheaf of holomorphic functions. $(k = \mathbb{C})$

Def A prevariety in the sense of Serre is a ringed space $(X, \mathcal{O}_X)$ s.t. every pt has an open nbhd isomorphic to an affine variety

To be a variety, we need an analogue of the Hausdorff condition.

Serre's theory is described in his paper Faisceaux Algébriques Cohérents (= Coherent Algebraic Sheaves).

is a predecessor to scheme theory, which we'll discuss next.

5 Sheaves

It is convenient to define sheaves of things more general than functions. Here is the definition

Def Given a top space X, a **presheaf** of sets, groups, ... is

an assignment

$$X \underset{\text{open}}{\supset} U \longmapsto \mathcal{F}(U) \text{ a set, group, ...}$$

To each pair $U \supseteq V$, a map, homomorphism...

$$\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$$

such that

$$\rho_{UU} = \mathrm{id}$$

and

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\rho_{UV}"] \arrow[dr, "\rho_{UW}"'] & \mathcal{F}(V) \arrow[d, "\rho_{VW}"] \\ & \mathcal{F}(W) \end{tikzcd}$$

commutes for any chain $U \supseteq V \supseteq W$

A slicker way to formulate this is to make $\mathrm{Open}(X)$ into a category. Then a presheaf is just a contravariant functor $\mathrm{Open}(X) \to \mathrm{Sets}$

In the previous examples ρ_{UV} is just restriction of functions

We will use the same notation.

Def A presheaf $\mathcal{F}$ is called a

sheaf if $\forall$ U and open cover $\{U_i\}$

Given $f_i \in \mathcal{F}(U_i)$ s.t.

$$f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$$

$\exists$ a unique $f \in \mathcal{F}(U)$ s.t

$$f|_{U_i} = f_i$$

All previous examples of sheaves are sheaves in this sense. We will treat more general examples next.

6. Affine Schemes

Thm/Def The set $X = \operatorname{Spec} R$ of prime ideals of R has a topology, also called the Zariski topology, with basic opens

$$D(f) = \{ p \in \operatorname{Spec} R \mid f \notin p \}$$

This possesses a sheaf of commutative rings $\mathcal{O}_X$ with

$$\mathcal{O}_X(D(f)) \cong R\left[\tfrac{1}{f}\right]$$

and $\rho_{D(f), D(gf)}$ the natural homomorphisms $R\left[\frac{1}{f}\right] \to R\left[\frac{1}{fg}\right]$

[See Hartshorne pp 70-72 for details]

The pair $(X, \mathcal{O}_X)$ is called the **affine scheme** associated to R.

It is worthwhile comparing with the classical story. If $X \subset \mathbb{A}^n_k$ is algebraic with k alg. closed. Let $R = \mathcal{O}(X)$.

Then Hilbert's Nullstellensatz we have a bijection between

$$X \longleftrightarrow \text{Max ideals of } R \subsetneq \operatorname{Spec} R.$$

The structure sheaf of $\operatorname{Spec} R$ can be pulled back to X and it coincides with the sheaf of regular functions constructed earlier.

So scheme theory vastly extends the classical picture.