1 Proj

In the last class we gave an example $\mathbb{P}^1_k$ of a non affine scheme defined by gluing. We want to generalize this example, but it's better to use a different construction, which we now explain.

Def A **graded** ring R, or precisely an $\mathbb{N}$-graded ring is a ring with a decomposition into abelian groups,

$$R = R_0 \oplus R_1 \oplus \cdots$$

such that $R_i \cdot R_j \subset R_{i+j}$.

Elements of R_i are called **homogeneous** of deg i.

An ideal $I \subseteq R$ is called **homogeneous** if it's generated by homogeneous elements.

Ex The polynomial ring $R = S[x_1, \ldots x_n]$ with the standard grading is of a graded ring.

Given a graded ring R,
$$R_+ = R_1 \oplus R_2 \oplus \cdots$$
is an ideal called the irrelevant ideal

Thm/Def Given a graded ring R. Let $\operatorname{Proj} R = \{$ homog. prime ideals $p^+ \subsetneq p \subseteq R$ $\}$

This can be made into a scheme where the basic open sets are

$$D_+(f) = \{ p \in \operatorname{Proj} R \mid f \notin p \}$$

where f homog. of deg d

$$\mathcal{O}_{\operatorname{Proj} R}(D_{\mathbb{P}}(f)) = \bigoplus_k R_{kd} \cdot \frac{1}{f^k}$$

$$= \underbrace{\left(R\left[\frac{1}{f}\right]\right)}_{\text{this has a natural } \mathbb{Z}\text{-grading}}{}_0$$

A pf can be found in Hartshorne pp 76-77. A key point is $\{ D_+(f) \cong \operatorname{Spec}(R[1/f])_0 \}$ a covering by affine schemes.
[Note Hartshorne uses $D(f)$ which conflicts with affine case]

<u>Def</u> Given a ring S define.

$$\mathbb{A}^n_S = \operatorname{Spec} S[x_1, \ldots x_n]$$

$$\mathbb{P}^n_S = \operatorname{Proj} S[x_0, \ldots x_n]$$

When $S = k$ is an alg. closed field the closed pts of the new $\mathbb{A}^n_k$ and $\mathbb{P}^n_k$ with classical affine or projective space i.e. pts of k^n or lines in k^{n+1}

So in this sense the new and old versions are the "same".

In general, set

$$U_i = D_{\mathbb{P}}(x_i) = \operatorname{Spec} S\left[\frac{x_0}{x_i}, \ldots \frac{x_n}{x_i}\right]$$
$$\cong \mathbb{A}^n_S$$

2 Closed Subschemes

Def A morphism $(f, f_\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of schemes is simply a morphism of locally ringed spaces.

Def A morphism $(i, i_\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of schemes is called a closed immersion if $i : X \to Y$ is homeomorphism between X and a closed subset $i(X) \subset Y$ and $i_\# : \mathcal{O}_Y \to i_* \mathcal{O}_X$ is a surjection (in the sheaf sense). We may as well take $X = i(X)$, and we refer to $(X, \mathcal{O}_X)$ as a closed subscheme of Y.

ex If $I \subset R$ is an ideal, the quotient map $R \to R/I$ induces a closed immersion $\operatorname{Spec}(R/I) \to \operatorname{Spec} R$. All closed immersions are locally of this form.

Def A **projective** scheme over R is a closed subscheme of some $\mathbb{P}^n_R$.

3 Coherent Sheaves on Schemes

Def If $(X, \mathcal{R})$ is a ringed space, a sheaf of $\mathcal{R}$-modules $\mathcal{M}$ is **quasicoherent** if every pt $x \in X$ has a nbhd U s.t. $\exists$ $\mathcal{O}_U^I \to \mathcal{O}_U^J \to \mathcal{M}|_U \to 0$. ($I, J$ may be infinite)

Thm If R is a ring, and M is an R-module, then

$$\tilde{M}(U) = M \otimes \mathcal{O}_{\operatorname{Spec} R}(U)$$

defines a quasicoherent sheaf Spec R. This gives an equivalence of categories

$$(R\text{-mod}) \xrightarrow{\sim} \left(\begin{matrix}\text{quasi Coherent} \\ \text{sheaves on Spec} R\end{matrix}\right)$$

NO CLASS Thurs March 7

For the pf using this definition see EGA I, Cor I.1.5.2 & Hartshorne Cor II.5.5

Cor A quasicoherent sheaf on a scheme is locally of the form $\tilde{M}$.

Def A scheme is noetherian if it has finite open cover of the form $\{U_i \cong \operatorname{Spec} R_i\}$ with R_i noetherian.

All the schemes in this class will have this property, so the condition should be assumed.

Def If X is a noetherian scheme, a quasicoherent sheaf $\mathcal{M}$ is coherent if it is locally of the form $\tilde{M}$, with $\tilde{M}$ finitely generated.

Next is a fundamental example.

Ex If $i : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ is a closed subscheme of a (noetherian) scheme. Then $\mathcal{I} = \ker[\mathcal{O}_X \to i_* \mathcal{O}_Y]$ is called the ideal sheaf of X. This is coherent.

A subexample.

Ex Let $Y \subset X$ be closed, then let
$$\mathcal{I}(U) = \{ f \in \mathcal{O}_X(U) \mid f|_Y \equiv 0 \}.$$
This makes $\mathcal{O}_Y$ into a reduced sheaf of rings i.e. $\mathcal{O}_Y(U)$ has no nilpotents. This is called the reduced subscheme structure of Y. There are others.

Ex Let k be a field (for simplicity). The complement of $U_0 \subset \mathbb{P}^n_k$ defines a closed set $H = V_{\mathbb{P}}(x_0)$.

The reduced subscheme structure makes $H \cong \mathbb{P}^{n-1}_k$. The ideal sheaf will be analyzed below. But first

Thm / Def Given a noetherian ring S and a fin. gen $\mathbb{Z}$-graded module M over $R = S[x_0, \ldots x_n]$, there exists a coherent sheaf $\tilde{\tilde{M}}$ over $\mathbb{P}^n_S$ such that

$$\tilde{\tilde{M}}(D(f)) \cong \left(M\left[\frac{1}{f}\right]\right)_0$$

for f homog. This gives an exact functor

$$\left(\begin{matrix}\text{f.g graded}\\ \text{mod}\end{matrix}\right) \longrightarrow \left(\begin{matrix}\text{coherent}\\ \text{sheaves on } \mathbb{P}^n\end{matrix}\right)$$

pf See Hartshorne pp 116 - 117 & EGA II 2.5 (They just write $\tilde{M}$)

NB This is not an equivalence, since for example $\tilde{\tilde{O}} = \widetilde{\widetilde{(S/S_+)}} = 0$

Given a graded R-mod M, let $M(d) = M$ as a module but new grading
$$M(d)_\ell = M_{d+\ell}$$

Def $\mathcal{O}_{\mathbb{P}^n_R}(d) = \widetilde{S(d)}$

$\mathcal{O}(1)$ is called the tautological "line bundle" or invertible sheaf.
(We explain what this means later.)

Previous Ex (cont) The ideal sheaf $\mathcal{I}$ of $H \subset \mathbb{P}^n_k$ is isomorphic to $\mathcal{O}_{\mathbb{P}^n}(-1)$

To see this, observe $\mathcal{I} = \tilde{I}$, where $I = (x_0) \cong S(-1)$ since it's the image of $S(-1) \xrightarrow{x_0} S$

Ex More generally if $Y \subset \mathbb{P}^n_k$ is a closed subscheme defined by a deg d homog poly, its ideal sheaf $\mathcal{I}_Y \cong \mathcal{O}_{\mathbb{P}^n}(-d)$

4 Locally free sheaves

Def A coherent sheaf $\mathcal{M}$ on a scheme X is **locally free** of rank r if $\exists$ a open affine cov $\{U_i\}$ s.t. $\mathcal{M}|_{U_i} \cong \widetilde{M}_i$, where M_i is free of rank r