1 Locally Free Sheaves

Last time, we defined

Def A sheaf of R-modules $\mathcal{M}$ on a ringed space (X, R) is locally free of rank n if $\exists$ an open cover $\{U_i\}$ s.t. $\mathcal{M}|_{U_i} \cong R_{U_i}^n$

Equivalently for schemes:

Def A coherent sheaf $\mathcal{M}$ on a scheme X is locally free of rank n if $\exists$ a open affine cover $\{U_i\}$ st. $\mathcal{M}|_{U_i} \cong \widetilde{M}_i$, where M_i is free of rank n

Prop If R is a noetherian ring, a finitely generated R-module M is projective $\Leftrightarrow$ $\widetilde{M}$ is locally free

[A proof can be found a p 109 of Bourbaki's Comm Alg.]

Without going into details, there are natural example of rings with non free projective (or loc free) modules. For non affine schemes, we have:

Thm (a) $\mathcal{O}_{\mathbb{P}^n_R}(d)$ is locally free of rank 1.

(b) If $d < 0$, then $H^0(\mathcal{O}_{\mathbb{P}^n}(d)) = 0$

and when $d > 0$, then $H^0(\mathcal{O}_{\mathbb{P}^n}(d)^\vee) = 0$

Here $M^\vee(U) = \mathrm{Hom}(M|_U, \mathcal{O}_U)$

(c) It's free iff $d = 0$ (or $n = 0$).

Pf Let $R = k$ be a field

If M is a graded module over $S = k[x_0, \dots x_n]$

then
$$\tilde{M}|_{U_i} = \widetilde{M[\tfrac{1}{x_i}]_0}$$
when $M[\frac{1}{x_i}]_0$ is viewed as an $R[\frac{x_0}{x_i}, \dots]$-mod

When $M = S(d)$, we see that $M[\frac{1}{x_i}]_0$ is a free module gen by $\frac{1}{x_i^d}$.

There $\mathcal{O}_{\mathbb{P}^n}(d)$ is locally free.

For (b), for $d > 0$, we have that $H^0(\mathcal{O}_{\mathbb{P}^n}(-d))$ consists of constant functions vanishing on a degree d hypersurface.

Therefore $H^0(\mathcal{O}_{\mathbb{P}^n}(-d)) = 0$.

For the 2nd half, we check

$$\mathcal{O}(d)^\vee \cong \mathcal{O}(-d)$$

Finally, if $\mathcal{O}(d) \cong \mathcal{O}$, then

$$H^0(\mathcal{O}(d)) = H^0(\mathcal{O}(d)^\vee) = k$$

Therefore (c) follows from (b). //

Def A real (or complex) vector bundle of rank n on a C^∞-manifold X is C^∞-map $\pi: V \to X$ such that there is an open cover $\{U_i\}$ with isomorphism

$$\begin{tikzcd} \pi^{-1}(U_i) \arrow[rr, "f_i", "\sim"'] \arrow[dr] & & U_i \times \mathbb{R}^n \ (\text{or } \mathbb{C}^n) \arrow[dl] \\ & U_i & \end{tikzcd}$$

s.t. f_i commutes with projections and such that $f_i \circ f_j^{-1}$ is linear on fibres. (This data is called a local trivialization)

Thm Given a real (resp. complex) vector bundle $\pi: V \to X$ of rank n over a manifold, the sheaf of sections

$$\mathcal{V}(U) = \{ \sigma : U \to \pi^{-1}(U) \mid \pi \circ \sigma = id \}$$

is naturally a locally free module of rank n over C^∞_X (resp $\mathbb{C} \otimes_{\mathbb{R}} C^\infty_X$). Furthermore every locally free sheaf arises this way.

A similar result holds for schemes, see Hartshorne p 128. For this reason, algebraic geometers tend to use the terms "locally free sheaf" and "vector bundle" interchangeably.

2 Nice schemes

Recall that we said all our schemes. Even so, there can be pretty wild. We various niceness conditions we can impose

Def A scheme $(X, \mathcal{O}_X)$ is

1) **reduced** if $\mathcal{O}_X(U)$ is reduced $\forall U$

$\Downarrow$

2) **integral** if $\mathcal{O}_X(U)$ is an integral domain $\forall U$

$\Downarrow$

3) **normal** if $\mathcal{O}_X(U)$ is an integrally closed domain $\forall U$

$\Downarrow$

4) **regular** if it's integral and all local rings $\mathcal{O}_{X,x}$ are regular.

Next, we need an analogue of the Hausdorff axiom. We start with the following simple characterization.

Thm A top space X is Hausdorff $\iff$ the diagonal $\Delta \subset X \times X$ is closed

For the analogue we need:

Thm The category of schemes admits products. This means that given X & Y $\exists$ $X \times Y$ with projection morphisms $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ which are universal in the sense that given solid arrows,

$$\begin{tikzcd} Z \arrow[rr] \arrow[d] \arrow[r, dashed] & X \times Y \arrow[r, "p_Y"] \arrow[dl, "p_X"] & Y \\ X & & \end{tikzcd}$$

$\exists$ a dotted arrow as above.

when $X = \operatorname{Spec} A$, $Y = B$,

$X \times Y = \operatorname{Spec}(A \otimes_{\mathbb{Z}} B)$

In general see Hartshorne II thm 3.3 for the construction.

From the universal prop, we get a morphism Δ_X called the diagonal

$$\begin{tikzcd} X \arrow[rr, "id"] \arrow[d, "id"'] \arrow[dr, "\Delta_X"] & & X \\ X & X \times X \arrow[l] \arrow[ur] & \end{tikzcd}$$

Def X is **seperated** if Δ_X is a closed immersion.

This is so basic that all schemes we will encounter satisfy this

Finally suppose we are given a morphism $X \to \operatorname{spec} k$, where k is field. This implies that the rings $\mathcal{O}_X(U)$ are k-algebras

Def A scheme $X \to \operatorname{Spec} k$ is of *finite type* if it possesses a finite cover $\{U_i \cong \operatorname{Spec} R_i\}$, where R_i are finitely generated k-algebras.

Finally we can *redefine*

Def $X \to \operatorname{spec} k$ an *algebraic variety* if it integral, separated and of finite type (Some people assume only that X is reduced).

3 Divisors

To motivate start with a compact Riemann surface X. Although there are no nonconstant holomorphic functions on X, There's plenty of nonconstant meromorphic functions on it. The collection of meromorphic functions forms a field $\mathbb{C}(X) \supsetneq \mathbb{C}$.

A basic problem going back to Riemann is to construct elements of $\mathbb{C}(X)$ with prescribed zeros and poles. We record this information by forming a finite formal sum $D = \sum n_p p$, $p \in X$, $n_p \in \mathbb{Z}$, called a divisor. Set

$$L(D) = \{ f \in \mathbb{C}(X) \mid \operatorname{ord}_p(f) \geq -n_p \ \forall p \}.$$

where $\operatorname{ord}_p(f) = m$ where

$$f(z) = a_m z^m + a_{m+1} z^{m+1} + \dots$$

is the Laurent expansion about p.

Riemann-Roch problem

What's $\dim_{\mathbb{C}} L(D)$?

We will see later that this dim is finite.

To help with we introduce the sheaf

$$\mathcal{O}_X(D)(\mathcal{U}) = \{ f \in \Gamma(X) \mid \forall p \in \mathcal{U}, \; f \geq -n_p \}$$

Then $L(D) = H^0(X, \mathcal{O}_X(D))$

Before saying more, we introduce the scheme theoretic version.

Let us assume from now on that X is a normal (noetherian, separated) scheme, e.g. a nonsingular alg. variety.

Def A *prime divisor* $D \subset X$ is a closed irreducible subset of codimension 1 (i.e. $\dim D = \dim X - 1$)

A Weil *divisor* is a finite formal linear combination $\sum n_i D_i$, where $D_i \subset X$ are prime divisors and $n_i \in \mathbb{Z}$.

By assumption, X is normal, and in particular integral. This implies that $\mathcal{O}_X(\mathcal{U})$ is an integral domain, and as $\mathcal{U}$ varies over nonempty open sets $\mathcal{O}_X(\mathcal{U})$ has a common field of fractions $K(X)$, called the function field of X. Elements of $K(X)$ are called rational functions on X.

A prime divisor D on X can be identified with a (nonclosed) point of X such that $\dim \mathcal{O}_{X,D} = 1$

A one dim integrally closed local ring is discrete valuation ring (see Atiyah-MacDonald). Therefore $\exists$ a function $\operatorname{ord}_D : K(X) \to \mathbb{Z} \cup \{\infty\}$ s.t.

1) $\operatorname{ord}_D(fg) = \operatorname{ord}_D f + \operatorname{ord}_D g$
2) $\operatorname{ord}_D(f+g) \geq \min(\operatorname{ord}_D f, \operatorname{ord}_D g)$
3) $\operatorname{ord}_D f = \infty \iff f = 0$
4) $\mathcal{O}_{X,D} = \{ f \mid \operatorname{ord}_D f \geq 0 \}$

Def Given a divisor
$$D = \sum n_i D_i,$$
Set $\mathcal{O}_X(D)(U) = \{ f \in k(X) \mid \mathrm{ord}_{D_i} f \geq -n_i \}$

NB Hartshorne uses $\mathcal{L}(D)$ for $\mathcal{O}_X(D)$, but most people use the last notation.

Def X is called *locally factorial* if all the local rings $\mathcal{O}_{X,p}$ are UFD.

We need 2 hard results from commutative Algebra (see Matsumura)

Thm (Auslander-Buchsbaum-Serre). A regular local ring is a UFD

Cor A regular scheme is locally factorial

Thm A noetherian domain is a UFD $\iff$ every height one prime is principal.

Thm If X is locally factorial, (e.g regular) then $\mathcal{O}_X(D)$ is locally free of rank one (= invertible sheaf = line bundle)

We need

lemma If $\mathcal{L}$ and $\mathcal{M}$ are invertible, then so are $\mathcal{L}^{-1} := \mathcal{L}^\vee$ and $\mathcal{L} \otimes \mathcal{M}$.

pf Exercise

pf of thm When D is prime, $\mathcal{O}_X(-D)$ is the ideal sheaf of D. This is invertible by the previous thm. In general write

$$D = \sum n_i D_i = \sum a_i A_i - \sum b_j B_j$$

with $a_i, b_j > 0$

Then $\mathcal{O}_X(D) = \bigotimes_i \mathcal{O}(A_i)^{a_i} \otimes \left(\bigotimes_j \mathcal{O}(B_j)^{b_j} \right)^{-1}$

is invertible by the lemma.

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