

Locally Free Sheaves

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Last time, we defined

Def A sheaf of R -modules, \mathcal{M} on a ringed space (X, \mathcal{R}) is **locally free** of rank n if \exists an open cover $\{U_i\}$ s.t. $\mathcal{M}|_{U_i} \cong \hat{\mathcal{R}}_{U_i}^n$.

Equivalently for schemes:

Def A coherent sheaf \mathcal{M} on a scheme X is **locally free** of rank n if \exists a open affine cover $\{U_i\}$ s.t. $\mathcal{M}|_{U_i} \cong \hat{\mathcal{M}}_i$, where \mathcal{M}_i is free of rank n .

Prop If R is a noetherian ring, a finitely generated R -module M is projective $\Leftrightarrow \hat{M}$ is locally free.

[A proof can be found on p 109 of Bourbaki's Comm Alg.]

Without going into details, there are natural examples of rings with non free projective (or loc free) modules. For non affine schemes, we have:

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Thm (a) $\mathcal{O}_{\mathbb{P}^n}(d)$ is locally free of rank 1.

(b) If $d < 0$, then $H^0(\mathcal{O}_{\mathbb{P}^n}(d)) = 0$

and when $d > 0$, then $H^0(\mathcal{O}_{\mathbb{P}^n}(d)) \neq 0$

Then $M^v(u) = \text{Hom}(M^u, \mathcal{O}_u)$

(c) It's free iff $d = 0$ (or $n = 0$).

Pf Let $R = k$ be a field

If M is a graded module over $S = k[x_0, \dots, x_n]$

Then $\widehat{M}|_{U_i} = \widehat{M[\frac{1}{x_i}]_0}$

when $M[\frac{1}{x_i}]_0$ is viewed as an $R[\frac{x_0}{x_i}, \dots]$ -mod

When $M = S(d)$, we see that $M[\frac{1}{x_i}]_0$ is a free module gen by $\frac{1}{x_i^d}$.

Thus $\mathcal{O}_{\mathbb{P}^n}(d)$ is locally free.

For (b), for $d > 0$, we have that

$H^0(\mathcal{O}_{\mathbb{P}^n}(-d))$ consists of constant functions vanishing on a degree d hyper-surface.

Therefore $H^0(\mathcal{O}_{\mathbb{P}^n}(-d)) = 0$.

For the 2nd half, we check
 $\mathcal{O}(d)^\vee \cong \mathcal{O}(-d)$

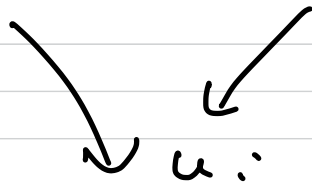
Finally, if $\mathcal{O}(d) \cong \mathcal{O}$, then

$$H^0(\mathcal{O}(d)) = H^0(\mathcal{O}(d)^\vee) = k$$

Therefore (c) follows from (b). //

Def A real (or complex) **vector bundle** of rank n on a C^∞ -manifold X is a C^∞ -map $\pi: V \rightarrow X$ such that there is an open cover $\{U_i\}$ with isomorphism

$$f_i: \pi^{-1}(U_i) \xrightarrow{\sim} U_i \times \mathbb{R}^n \text{ (or } \mathbb{C}^n)$$



s.t. f_i commutes with projections and such that $f_i \circ f_j^{-1}$ is linear on fibres. (This data is called a **local trivialization**)

Thm Given a real (resp. complex) vector bundle $\pi: V \rightarrow X$ of rank n over a manifold, the sheaf of sections

$$\mathcal{V}(U) = \{ \sigma: U \rightarrow \pi^{-1}(U) \mid \pi \circ \sigma = \text{id} \}$$

is naturally a locally free module of rank n over $C^\infty_\mathbb{R}$ (resp $\mathbb{C} \otimes_{\mathbb{R}} C^\infty_\mathbb{R}$)

Furthermore every locally free sheaf arises this way.

A similar result holds for schemes, see Hartshorne p 128. For this reason, algebraic geometers tend to use the terms "locally free sheaf" and "vector bundle" interchangeably.

2 Nice schemes

Recall that we said all our schemes. Even so, there can be pretty wild. We various niceness conditions we can impose

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Def A scheme (X, \mathcal{O}_X) is

- 1) **reduced** if $\mathcal{O}_X(U)$ is reduced $\forall U$
- \Downarrow
- 2) **integral** if $\mathcal{O}_X(U)$ is an integral domain $\forall U$
- \Downarrow
- 3) **normal** if $\mathcal{O}_X(U)$ is an integrally closed domain $\forall U$
- \Downarrow
- 4) **regular** if its integral all all local rings $\mathcal{O}_{X, x}$ are regular.

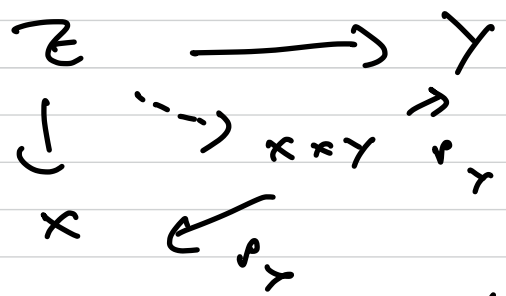
Next, we need an analogue of the Hausdorff axiom. We start with the following simple characterization.

Thm A top space X is Hausdorff \iff the diagonal $\Delta \subset X \times X$ is closed

For the analogue we need:

Thm The category of schemes admits products. This means that given X & Y \exists $X \times Y$ with projection morphisms $p_X: X \times Y \rightarrow X$ and $p_Y: X \times Y \rightarrow Y$ which are universal in the sense that given

Solid arrows,

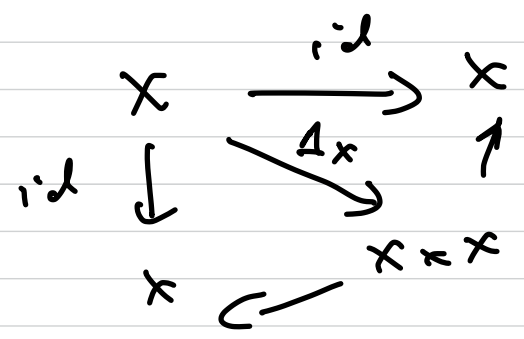


∫ a dotted arrow as above.

When $X = \text{Spec } A$, $Y = B$,
 $X \times Y = \text{Spec } (A \otimes_{\mathbb{Z}} B)$

In general see Hartshorne II thm 3.3 for the construction.

From the universal prop, we get a morphism Δ_X called the diagonal



Def X is separated if Δ_X is a closed immersion.

This is so basic that all schemes we will encounter satisfy this

Finally suppose we are given a morphism $X \rightarrow \text{Spec } k$, where k is a field. This implies that the rings $\mathcal{O}_x(U)$ are k -algebras

Def A scheme $X \rightarrow \text{Spec } k$ is of finite type if it possesses a finite cover $\{U_i \cong \text{Spec } R_i\}$, where R_i are finitely generated k -algebras.

Finally we can redefine

Def $X \rightarrow \text{Spec } k$ an algebraic variety if it is integral, separated and of finite type (some people assume only that X is reduced).

3 Divisors

To motivate start with a compact Riemann surface X . Although there are no nonconstant holomorphic functions on X ,

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There plenty of non-constant meromorphic functions on \mathbb{C} . The collection of meromorphic functions forms a field $\mathbb{C}(x) \supsetneq \mathbb{C}$.

A basic problem going back to Riemann is to construct elements of $\mathbb{C}(x)$ with prescribed zeros and poles. We record this information by forming a finite formal sum $D = \sum n_p p$, $p \in x$

$n_p \in \mathbb{Z}$, called a **divisor**. Set

$$L(D) = \{ f \in \mathbb{C}(x) \mid \text{ord}_p(f) \geq -n_p \forall p \}$$

where $\text{ord}_p(f) = m$ where

$$f(z) = a_m z^m + a_{m+1} z^{m+1} + \dots$$

is the Laurent expansion about p .

Riemann-Roch problem

What's $\dim_{\mathbb{C}} L(D)$?

We will see later that this dim is finite.

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To help with we introduce the sheaf

$$\mathcal{O}_X(D)(U) = \left\{ f \in \mathbb{C}(X) \mid \forall p \in U, f \geq -n_p \right\}$$

Then $L(D) = H^0(X, \mathcal{O}_X(D))$

Before saying more, we introduce the scheme theoretic version.

Let us assume from now on that X is a normal (noetherian, separated) scheme, e.g. a nonsingular alg. variety.

Def A prime divisor $D \subset X$

is a closed irreducible subset of codimension 1 (i.e. $\dim D = \dim X - 1$)
Weil

A n -divisor is a finite formal linear combination $\sum n_i D_i$, where $D_i \subset X$ are prime divisors and $n_i \in \mathbb{Z}$.

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B_2 assumption, X is normal, and in particular integral. This implies that $\mathcal{O}_x(\mathcal{U})$ is an integral domain, and as \mathcal{U} varies over nonempty open sets $\mathcal{O}_x(\mathcal{U})$ has a common field of fractions $K(X)$, called the **function field** of X . Elements of $K(X)$ are called **rational functions** on X .

A prime divisor D on X can be identified with a (nonclosed) point of X such that $\dim \mathcal{O}_{x,D} = 1$

A one dim integrally closed local ring is discrete valuation ring (see Atiyah-H-MacDonald). Therefore

\exists a function $\text{ord}_D: K(X) \rightarrow \mathbb{Z} \cup \{\infty\}$

- s.t.
- 1) $\text{ord}_D(fg) = \text{ord}_D f + \text{ord}_D g$
 - 2) $\text{ord}_D(f+g) \geq \min(\text{ord}_D f, \text{ord}_D g)$
 - 3) $\text{ord}_D f = \infty \Leftrightarrow f = 0$
 - 4) $\mathcal{O}_{x,D} = \{f \mid \text{ord}_D f \geq 0\}$

Def Given a divisor

$$D = \sum n_i D_i,$$

Set $\mathcal{O}_x(D)(u) = \{f \in k(x) \mid \text{ord}_{D_i} f \geq -n_i\}$

NB Hartshorne uses $\mathcal{L}(D)$ for $\mathcal{O}_x(D)$, but most people use the last notation.

Def X is called **locally factorial**

if all the local rings $\mathcal{O}_{x,p}$ are

VFD.

We need 2 hard results from commutative Algebra (see Matsum)

Thm (Auslander - Buchsbaum - Serre)

A regular local ring is a VFD

Cor A regular scheme is locally factorial

Thm A noetherian domain is a VFD \Leftrightarrow

every height one prime is principal.

Thm If X is locally factorial,
 (e.g. regular) then $\mathcal{O}_X(D)$ is locally
 free of rank one (= invertible sheaf
 = line bundle)

We need

lemma If f and M are invertible, then
 so are $f^{-1} := f^\vee$ and $f \otimes M$.

pf Exercise

pf of thm When D is prime, $\mathcal{O}_X(-D)$ is
 the ideal sheaf of D . This is invertible
 by the previous thm. In general write

$$D = \sum_i n_i D_i = \sum_i a_i A_i - \sum_j b_j B_j$$

with $a_i, b_j > 0$

$$\text{Then } \mathcal{O}_X(D) = \left(\bigotimes_i \mathcal{O}(A_i)^{a_i} \right) \otimes \left(\bigotimes_j \mathcal{O}(B_j)^{b_j} \right)^{-1}$$

is invertible by the lemma.

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