Divisor Class Group

Let X be a normal separated scheme.

lemma/def Given a nonzero rational function $f \in K(X)^*$, the sum

$$(f) = \sum_D \operatorname{ord}_D(f) D$$

is finite, therefore it defines a divisor. A divisor of this form is called principal.

The set $\operatorname{Div}(X)$ of divisors forms an abelian group in an obvious way.

lemma/def The subset of principal divisors $\operatorname{Princ}(X) \subset \operatorname{Div}(X)$ forms a subgroup. The quotient $\operatorname{Cl}(X) = \dfrac{\operatorname{Div}(X)}{\operatorname{Princ}(X)}$ is called the divisor class group.

pf Since ord_D is a valuation

$$(fg) = \sum \operatorname{ord}_D(fg) D = (f) + (g)$$

$$(f^{-1}) = \sum \operatorname{ord}_D(f^{-1}) D = -(f).$$

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Def Two divisors D, D' are called **linearly equivalent** ($D \sim D'$) if $D - D'$ is principal. i.e. if they define the same element of $Cl(X)$.

Thm $\mathcal{O}_X(D) \cong \mathcal{O}_X(D') \iff D \sim D'$

pf In one direction. If $D' - D = (g)$

Then
$$\mathcal{O}(D)(U) \longrightarrow \mathcal{O}(D')(U)$$
$$f \longmapsto f \cdot g$$
is an isomorphism. //

Thm/Def The set of isomorphism classes of line bundles forms an abelian group $\mathrm{Pic}(X)$ called the **Picard group**. The group operation is given by $\otimes$.

Suppose that X is locally factorial. Then $\mathcal{O}_X(D)$ is a line bundle.

Lemma If X is loc. factorial, then

$$\mathcal{O}(D) \otimes \mathcal{O}(D') \cong \mathcal{O}(D+D')$$

Consequently, we have a homomorphism

$$\mathrm{Cl}(X) \longrightarrow \mathrm{Pic}(X)$$

Thm If X is locally factorial then

$$\mathrm{Cl}(X) \cong \mathrm{Pic}(X)$$

We'll give the proof shortly

2 1st Čech cohomology

Def Given $\mathcal{F} \in \mathrm{Ab}(X)$ and an open cover $\mathcal{U} = \{U_i\}$,

a (Čech) 1-cocycle is a collection
$f_{ij} \in \mathcal{F}(\mathcal{U}_{ij})$ s.t.

$$\begin{cases} f_{ik} = f_{ij} + f_{jk} & \text{on } \mathcal{U}_{ijk} \\ f_{ii} = 0 \end{cases}$$

where $\mathcal{U}_{ij} = \mathcal{U}_i \cap \mathcal{U}_j$ etc.

A 1-coboundary is cocycle f_{ij} s.t.
$f_{ij} = g_i - g_j$, where $g_i \in \mathcal{F}(\mathcal{U}_i)$

The 1st Čech gps

$$\check{H}^1(\mathcal{U}, \mathcal{F}) = \frac{\text{1-cocycles}}{\text{1-coboundaries}}$$

$$\check{H}^1(X, \mathcal{F}) = \lim_{\substack{\longrightarrow \\ \mathcal{U} \text{ under} \\ \text{refinement}}} \check{H}^1(\mathcal{U}, \mathcal{F})$$

Thm $\quad \check{H}^1(X, \mathcal{F}) \cong H^1(X, \mathcal{F})$

Thm Let $(X, \mathcal{O}_X)$ be a scheme, then

$$\mathrm{Pic}(X) \cong H^1(X, \mathcal{O}_X^*)$$

where $\mathcal{O}_X^*$ is sheaf units.

Sketch Given $\mathcal{L} \in \mathrm{Pic}(X)$. We can find a cover $\{U_i\}$ and isomorphisms $\varphi_i : \mathcal{L}|_{U_i} \xrightarrow{\sim} \mathcal{O}_{U_i}$

On U_{ij},

$$\varphi_i \circ \varphi_j^{-1} : \mathcal{O}_{U_{ij}} \xrightarrow{\sim} \mathcal{O}_{U_{ij}}$$

is given by multiplication by a unit $f_{ij} \in \mathcal{O}^*(U_{ij})$. This is a 1-cocycle with values in $\mathcal{O}_X^*$. Multiplying f_{ij} by a coboundary corresponds to the same line bundle $\mathcal{L}$ with a different choice of isomorphisms $\varphi_i' : \mathcal{L}|_{U_i} \to \mathcal{O}_{U_i}$

This gives a homomorphism

$$\mathrm{Pic}(X) \longrightarrow \check{H}^1(X, \mathcal{O}_X^*)$$

Given an element of $\check{H}^1(X, \mathcal{O}_X^*)$ it can be realized by a 1-cocycle $f_{ij} \in \mathcal{O}_X^*(U_{ij})$. We can use this to build a line bundle by gluing $\mathcal{O}_{U_i}$ to $\mathcal{O}_{U_j}$ using f_{ij}. This gives an inverse

$$\check{H}^1(X, \mathcal{O}^*) \longrightarrow \mathrm{Pic}(X) \qquad //$$

3 Cartier Divisors

Let X be a normal sep. scheme as before.

Def A Cartier divisor is a global section of $K_X^*/\mathcal{O}_X^*$, where K_X^* is sheaf $U \mapsto K_X(X)^*$. Let $\mathrm{CaDiv}(X)$ denote the group of Cartier divisors.

Concretely, a Cartier divisor is given by a collection $f_i \in k(U_i)$ s.t. $f_i/f_j \in \mathcal{O}(U_{ij})^*$ for some open cover $\{U_i\}$ of X. This implies
$$\text{ord}_D f_i = \text{ord}_D f_j$$
whenever $D \cap U_{ij} \neq \emptyset$

The previous notion of divisor are called Weil divisors to distinguish them.

Given $\{f_i\}$, we associate the (Weil) divisor
$$\sum \text{ord}_D (f_i) D$$
This is easily seen to give a homomorphism
$$\text{Ca Div}(X) \to \text{Div}(X)$$

Thm If X is loc. factorial, then
$$\text{Ca Div}(X) \cong \text{Div}(X)$$

Pf See Hartshorne II prop 6.11

Thm If X is loc. factorial,

$$Cl(X) \cong Pic(X)$$

Pf. Consider the exact sequence

$$1 \to \mathcal{O}_X^* \to K_X^* \to K_X^*/\mathcal{O}_X^* \to 1$$

This gives

$$\begin{tikzcd}
H^0(X, K_X^*) \arrow[r] \arrow[d] & CaDiv(X) \arrow[r] \arrow[d] & Pic(X) \arrow[r] & H^1(X, K_X^*) \arrow[d, equal] \\
Princ(X) \arrow[r] & Div(X) \arrow[ur] & & 0
\end{tikzcd}$$

$\because K_X^*$ flasque

which implies the thm. //