Abelian Categories

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1 Abelian categories

A category $\mathcal{A}$ is called abelian if it behaves like the category of abelian groups *Ab*, or generally modules over a ring. The usual axiomatizations tend to be a bit too concise. So instead let's write down a long list of conditions on category $\mathcal{A}$, which hold in these examples.

A1. $Hom_{\mathcal{A}}(M, N)$ is an abelian group for every pair of objects M, N.

A2. Composition satisfies $f \circ (g + h) = f \circ g + f \circ h$ whenever both sides are defined. Similarly, $(g + h) \circ f = g \circ f + h \circ f$ when this makes sense.

A3. There is a zero object satisfying $Hom_{\mathcal{A}}(0, M) = Hom_{\mathcal{A}}(M, 0) = 0$ for all M.

A4. For any pair of objects M, N we can form a direct sum, characterized up to isomorphism by $Hom(M \oplus N, T) = Hom(M, T) \oplus Hom(N, T)$ and $Hom(T, M \oplus N) = Hom(T, M) \oplus Hom(T, N)$

A5. Given a morphism $f : M \to N$, we can form an object $\ker f$ with a morphism $\ker f \to M$ characterized by

$$Hom(T, \ker f) = \ker[Hom(T, M) \to Hom(T, N)]$$

A6. Given $f : M \to N$, we can form an object $\operatorname{coker} f$ with a morphism $M \to \operatorname{coker} f$, characterized by

$$Hom(\operatorname{coker} f, T) = \ker[Hom(N, T) \to Hom(M, T)]$$

A7. Given $f : M \to N$, there exists an object $\operatorname{im} f$ with morphisms $M \to \operatorname{im} f$ and $\operatorname{im} f \to N$ such that their composition is f. We also require that $\operatorname{im} f$ is both $\operatorname{coker}(\ker f \to M)$ and $\ker(N \to \operatorname{coker} f)$. (A bit more precisely, these are canonically isomorphic.)

A category is called *additive* if A1-A4 hold, and it is called *abelian* if they all hold. The last axiom is the hardest to fathom. It is trying to capture the idea that in *Ab*, f can be factored through a surjective homomorphism

$M \to \operatorname{im} f$ followed by an injective homomorphism $\operatorname{im} f \to N$. Since injectivity and surjectivity are not categorical notions, we replace them by saying that they are kernels or cokernels. To appreciate further subtleties, see example 5. In an abelian category, since we have kernels and images, we have a notion of exact sequence.

Example 2. *The category Mod_R of modules over a commutative ring is an abelian category.*

Example 3. *The category of finitely generated modules over a noetherian ring is abelian. In particular, this applies to finitely generated abelian groups.*

Example 4. *The category of free abelian groups is additive but not abelian, because cokernels need not exist.*

Example 5. *The category of Hausdorff topological abelian groups and continuous homomorphisms satisfies A1-A6. The operations are the usual ones except for the cokernel. The cokernel of $f : M \to N$ in this category is the quotient $N/\overline{f(M)}$. However, if $f(M)$ is not closed, the map from $\operatorname{coker}(\ker f \to M) = M/\ker f$ to $\ker(N \to \operatorname{coker} f) = \overline{f(M)}$ is not an isomorphism. So A7 fails.*

Here is a simple yet powerful observation.

Proposition 6. *If A is abelian (resp. additive), then so is the opposite category A^{op}. This has the same objects as A but arrows are reversed, so that $Hom_{A^{op}}(N, M) = Hom_A(M, N)$.*

Proof. The axioms are self dual. □

Therefore

Example 7. *Mod_R^{op} is an abelian category. (NB: This should not be confused with $Mod_{R^{op}}$.)*

8 Sheaves

Given a topological space X, let $PAb(X)$ and $Ab(X)$ denote the category of presheaves and sheaves of abelian groups on X. Recall that $Ab(X) \subset PAb(X)$, and we have a functor $+$ in the opposite direction (it's the right adjoint).

Proposition 9. *The categories $PAb(X)$ and $Ab(X)$ are abelian.*

In outline:

A1 If $\eta, \xi\ \mathcal{F} \to \mathcal{G}$ are morphisms in $PSh(X)$, then

$$(\eta + \xi)_U = \eta_U + \xi_U$$

defines an abelian group structure on $Hom(\mathcal{F}, \mathcal{G})$. This also works for sheaves because $Ab(X) \subset PAb(X)$ is a full subcategory.

A2 Easy exercise.

A3 The zero sheaf is

$$0(U) = 0$$

A4

$$\mathcal{F} \oplus \mathcal{G}(U) = \mathcal{F}(U) \oplus \mathcal{G}(U)$$

A5 If $\eta : \mathcal{F} \to \mathcal{G}$ is a morphism in $PAb(X)$, the kernel in this category is

$$pker\, \eta(U) = \ker \eta_U$$

If η is a morphism in $Ab(X)$, in class we proved that

$$\ker \eta = pker\, \eta$$

is also a sheaf.

A6 If $\eta : \mathcal{F} \to \mathcal{G}$ is a morphism in $PAb(X)$, the cokernel in this category is

$$pcoker\, \eta(U) = \operatorname{coker} \eta_U$$

If η is a morphism in $Ab(X)$,

$$\operatorname{coker} \eta = (pcoker\, \eta)^+$$

is the cokernel.

A7 If $\eta : \mathcal{F} \to \mathcal{G}$ is a morphism in $PAb(X)$, the image in this category is

$$pim, \eta(U) = \operatorname{im} \eta_U$$

If η is a morphism in $Ab(X)$,

$$\operatorname{im} \eta = (pim\, \eta)^+$$

is the image.

10 Injectives

Ever since the work of Cartan-Eilenberg [CE], the standard approach to homological algebra is via injective and projective resolutions. To explain what this means, let's start with something elementary. An abelian group M is called *divisible* if for any $m \in M$ and nonzero integer $n \in \mathbb{Z}$, there exists $m' \in M$ such that $nm' = m$ (morally this says m/n exists). For example $\mathbb{Q}, \mathbb{R}$ and $\mathbb{Q}/\mathbb{Z}$ are divisible but no nonzero finitely generated group is. It's clear that any homomorphism from $\mathbb{Z} \supset n\mathbb{Z} \to M$ extends to $\mathbb{Z} \to M$, when M is divisible. This suggest the following.

Definition 11. *An object M of an abelian category is injective, if given a subobject $A \subset B$ any morphism from a subobject $A \to M$ extends to a morphism $B \to M$.*

Here's what makes them special.

Lemma 12. *If A is injective, then any exact sequence*

$$0 \to A \to B \to C \to 0$$

splits.

Proof. The identity $id : A \to A$ extends to a morphism $B \to A$, and this is a splitting. ☐

Corollary 13. *If $F : \mathcal{A} \to \mathcal{B}$ is a left exact functor, with the above notation and assumptions,*

$$0 \to F(A) \to F(B) \to F(C) \to 0$$

is exact.

For an abelian group, it is not hard to show that it's injective if and only if it is divisible. So most groups or more generally modules are not injective, but it's possible to embed an arbitrary module into an injective one, and this is the starting point for the derived functors approach to homological algebra. This is not true for an arbitrary abelian category, e.g. if fails for the category of finitely generated abelian groups. However, a sufficient condition was given by Grothendieck [G].

Definition 14. *An abelian category $\mathcal{A}$ is called a Grothendieck category if additionally it satisfies.*

- *A8 It possess arbitrary (infinite) direct sums, and these operations preserve exactness.*
- *A9 It possess arbitrary direct limits (= filtered colimits), and these operations preserve exactness.*
- *A10 There is an object R, called a generator, such that every object A is a quotient of $\bigoplus_I R$, for some set I depending on A.*

Theorem 15 (Grothendieck)**.** *If $\mathcal{A}$ is a Grothendieck category, then any object embeds into an injective object.*

Of course, Grothendieck didn't call it a Grothendieck category. The precise reference is théorème 1.10.1 in [G]. He actually shows that there is a functor $M : \mathcal{A} \to \mathcal{A}$ such that $M(A)$ is injective, and there is a monomorphism $A \to M(A)$. $M(A)$ is constructed as a certain direct limit indexed by an ordinal number. In particular, the existence uses the axiom of choice in the form of the well ordering theorem, so it's not constructive. Also $M(A)$ is very big!

Here are two key examples.

Example 16. *The category of modules over a ring R is a Grothendieck category (This is also true if the ring is noncommutative. In this case, one should specify left or right modules.). For the generator, we can take R itself.*

Example 17. *The categories $PAb(X)$ and $Ab(X)$ are Grothendieck. In the last case (which is what we mostly care about), the sheafification of*

$$\mathbb{Z}(U) = \mathbb{Z}$$

is a generator.

In particular, it follows that

Corollary 18. *A module or sheaf of abelian groups embeds into an injective object.*

This can be proved more directly in these 2 cases using Zorn's lemma – and it usually is in most textbooks – but the main complaint above still applies.

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By repeated application of Grothendieck's theorem, or one of the above special cases, we obtain

Corollary 20. *For every object A of a Grothendieck category, we can find an exact sequence*

$$0 \to A \to I^0 \to I^1 \dots$$

where each I^k is injective. This is called an injective resolution.

This is the starting point for constructing right derived functors which I won't say anything about other than sheaf cohomology $H^i(X, A)$ is usually defined as the cohomology of the complex

$$\Gamma(X, I^0) \to \Gamma(X, I^1) \to \dots$$

Injective resolutions are not unique on the nose, but they are unique up to homotopy equivalence, and that's enough to guarantee that $H^i(X, A)$ are well defined in this approach. For a fairly modern introduction to all of this, my favourite book is Weibel's [W].

References

[CE] H. Cartan, S. Eilenberg, Homological Algebra, Princeton U. Press 1955

[G] A. Grothendieck, Sur quelques points d'algèbre homologique, Tohoku Math J 1957

[W] C. Weibel, An introduction to homological algebra, Cambridge U. Press 1994