Chapter 1

Affine Geometry

1.1 Algebraic sets

Let k be a field or possibly a commutative ring. We write $\mathbb{A}_k^n = k^n$, and call this n dimensional affine space[1]over k. Let

$$k[x_1, \ldots x_n] = \left\{ \sum c_{i_1 \ldots i_n} x_1^{i_1} \ldots x_n^{i_n} \mid c_{i_1 \ldots i_n} \in k \right\}$$

be the polynomial ring. Given $a = (a_i) \in \mathbb{A}^n$, we can replace x_i by $a_i \in k$ in f to obtain an element denoted by $f(a)$ or $ev_a(f)$, depending on our mood. A polynomial f gives a function $ev(f) : \mathbb{A}_k^n \to k$ defined by $a \mapsto ev_a(f)$.

Given $f \in k[x_1, \ldots x_n]$, or a set of polynomials S, define its zero set by

$$V(f) = \{a \in \mathbb{A}_k^n \mid f(a) = 0\}$$

$$V(S) = \bigcap_{f \in S} V(f)$$

It is easy to see that $V(S) = V(I)$, where I is the ideal generated by S. A subset is called *algebraic* if equals $V(I)$ for some I. For a general field, it is very hard to determine whether or not this is empty. For a ring, it's actually impossible:

Theorem 1.1.1 (Matjisevich)**.** *Hilbert's 10th problem has a negative solution. In other words, when $k = \mathbb{Z}$, an algorithm for deciding whether or not $V(S) = \emptyset$ does not (and cannot) exist.*

From now on (until we say otherwise), assume that k is an algebraically closed field. Then the above problem is easy.

Theorem 1.1.2 (The weak Hilbert Nullstellensatz)**.** *If k is algebraically closed and $I \subsetneq k[x_1, \ldots x_n]$ is a proper ideal, then $V(I)$ is nonempty.*

[1]You might wonder why this isn't simply called a vector space. The reason is that for an affine space, 0 plays no special role. So for example, an automorphism of $\mathbb{A}_k^n$ is not required to fix 0.

This says that

$$V(f_1,\dots,f_N)=\emptyset \Leftrightarrow$$
$$(f_1,\dots,f_N)=1 \Leftrightarrow$$
$$\exists g_i\in k[x_1,\dots x_n], \sum g_i f_i = 1$$

To see that it really can made into an algorithm, we need an effective form of the Nullstellsatz found by Grete Hermann in the 1920's.

Theorem 1.1.3 (Hermann)**.** *In the above situation there exists an effective upper bound on* $\deg g_i$ *in terms of* $D=\max\{\deg f_i\}$ *and* n*.*

Note that her bound was on the order of D^{2^n}, which is pretty bad. Finding better ones, is actually an area of current research.[2]

While we're on the subject, we should recall the stronger form the Nullstellensatz. Given a subset $X\subset \mathbb{A}^n_k$, let

$$\mathcal{I}(X)=\{f\in k[x_1,\dots x_n]\mid f(a)=0,\forall a\in X\}$$

This is a special kind of ideal of $k[x_1,\dots x_n]$ called a radical ideal. This means $f\in I$ if $f^N\in I$.

Theorem 1.1.4 (Hilbert's Nullstellensatz)**.** *If* k *is algebraically closed, then*

1. $\mathcal{I}(V(I))=\sqrt{I}=\{f\mid \exists N, f^N\in I\}$.
2. $V(\mathcal{I}(X))$ *is the smallest algebraic set containing* X*.*
3. *The two operations yield inverse bijections as indicated.*

$$\{\textit{algebraic subsets of } \mathbb{A}^n\} \underset{V}{\overset{\mathcal{I}}{\rightleftarrows}} \{\textit{radical ideals in } k[x_1,\dots x_n]\}$$

Theorem/Def 1.1.5. *Let* k *be a field, then*

1. $V(0)=\mathbb{A}^n_k$, *and* $V((1))=\emptyset$.
2. *If* I *and* J *are ideals,* $V(I\cap J)=V(I)\cup V(J)$*.*
3. *If* $\{I_a\}$ *is a family of ideals, then*

$$V(\sum I_a)=\bigcap V(I_a)$$

Therefore the collection of algebraic sets forms the collection of closed subsets for a topology called the **Zariski topology***.*

[2] For a somewhat recent example, see Kollár, Sharp Effective Nullstellensatz, JAMS (1988)

When $k = \mathbb{C}$, $\mathbb{A}^n_k = \mathbb{C}^n$ has another topology coming from the Euclidean metric. We refer to this as the classical or strong topology. If $f \in k[x_1, \ldots x_n]$, set

$$D(f) = \mathbb{A}^n_k - V(f)$$

These sets form a basis for the Zariski topology. This means, in particular, that any open set is a union of sets of this form. Its clear from the definition that the closed subsets of $\mathbb{A}^1_k$ are either finite or all of $\mathbb{A}^1$. This means that it is very different from the topologies coming from metrics that one encounters in analysis. In particular, it is never Hausdorff when k is infinite. Another difference is the Noetherian property:

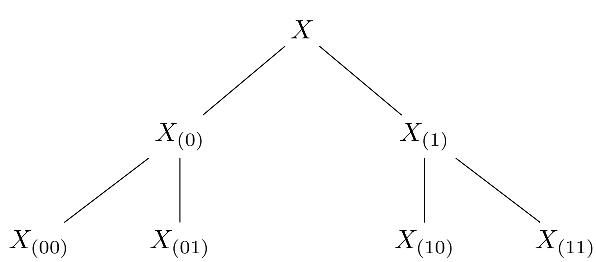
Lemma 1.1.6. *The Zariski topology is Noetherian, which means that there are no infinite strictly decreasing chains of closed sets* $\mathbb{A}^n = Z_0 \supsetneqq Z_1 \supsetneqq Z_2 \ldots$.

Proof. If there were such a chain, then by the Nullstellensatz $\mathcal{I}(Z_0) \subsetneqq \mathcal{I}(Z_1) \subsetneqq \ldots$ would be an infinite increasing chain of ideals. This would contradict the Noetherianness of $k[x_1, \ldots x_n]$. □

We say that X is *reducible* if it is a union of two proper closed sets. Otherwise X is called *irreducible*. The unique factorization property for polynomials can be generalized as follows.

Theorem 1.1.7. *Any Noetherian space X can be expressed as a union of $X = X_1 \cup X_2 \cup \ldots X_n$ of irreducible closed sets, where no X_i is contained in an X_j. This is unique up to reordering.*

Proof. If X is irreducible, there is nothing to prove. Suppose X is reducible, then we can write $X = X_{(0)} \cup X_{(1)}$ where $X_{(i)}$ are proper and closed. Repeat this for each $X_{(i)}$, and continue doing this. Let's represent this as a tree:



By the Noetherian property, we can't continue this forever. Thus the tree must be finite (we're using König's lemma that an infinite binary tree contains an infinite path). The "leaves", i.e. the lowest elements give the X_i.

Suppose we have another decomposition, $X = X'_1 \cup X'_2 \cup \ldots X'_m$. Then

$$X'_i = (X_1 \cap X'_i) \cup (X_2 \cap X'_i) \cup \ldots$$

Since the left side is irreducible, we must have $X'_i = X_j \cap X'_i$ for some j. So that $X'_i \subseteq X_j$. By symmetry, $X_j \subseteq X'_\ell$. Therefore $X'_i = X'_\ell$ by assumption, and this forces $X'_i = X_j$. This proves

$$\{X'_1, X'_2, \ldots\} \subseteq \{X_1, X_2, \ldots\}$$

We get the opposite inclusion by symmetry. □

The X_i in the theorem are called the irreducible components. An irreducible closed subset of some $\mathbb{A}^n_k$ is called an *affine algebraic variety.* There is a natural characterization in terms of ideals.

Lemma 1.1.8. *$X \subseteq \mathbb{A}^n_k$ is irreducible iff $\mathcal{I}(X)$ is a prime ideal.*

Proof of one direction. Suppose that $I = \mathcal{I}(X)$ is prime but that $X = X_1 \cup X_2$, where $X_i \subsetneqq X$ are proper closed. Then $I \subsetneqq I_i = \mathcal{I}(X_i)$. Therefore, there exists $f_i \in I_i$ such that $f_i \notin I$. But $f_1 f_2 \in I_1 \cap I_2 = I$. Therefore one of the $f_i \in I$ which is a contradiction.

□

In terms of ideal theory, the irreducible components of X correspond to the minimal primes of $I = \mathcal{I}(X)$. That is primes ideals p containing I and minimal with respect to this property.

1.2 Regular maps

One of the key realizations from the 1950's onwards, is that in mathematics it's not enough to know what the objects for a particular theory are, we also have to know what the maps are. In group theory, the right maps to consider are homomorphisms, in topology, they are continuous maps and so on. Here the right notion is called a *regular map* or *morphism.* This is a map defined by polynomials. Assume for simplicity that k is an algebraically closed. In particular k is an infinite field. This allows us to identify a polynomial with the function it defines. By definition, a morphism $f : \mathbb{A}^n_k \to \mathbb{A}^m_k$ is a map of the form

$$f(a_1, \ldots, a_n) = (F_1(a_1, \ldots, a_n), \ldots, F_m(a_1, \ldots, a_n)) \tag{1.1}$$

where $F_i \in k[x_1, \ldots x_n]$. In general, given algebraic sets $X \subset \mathbb{A}^n_k, Y \subset \mathbb{A}^m_k$, a map $f : X \to Y$ is called a morphism or regular if it can be defined by the same formula (1.1). A morphism $f : X \to Y$ is an isomorphism if it is bijective and f^{-1} is also a morphism.

Lemma 1.2.1. *A regular map $f : X \to Y$ is continuous with respect to the Zariski topologies.*

Proof. It's enough to know that preimages of closed sets are closed (why?), but this is obvious. □

If X is algebraic, a morphism $f : X \to \mathbb{A}^1_k$ is called a *regular function* on X. Let $\mathcal{O}(X)$ denote the set of regular functions. Since $\mathbb{A}^1_k = k$ is a commutative ring, $\mathcal{O}(X)$ becomes a commutative ring under pointwise addition and multiplication. This is called the *coordinate ring* of X.

Lemma 1.2.2. *If $X \subset \mathbb{A}^n_k$, $\mathcal{O}(X) \cong k[x_1, \ldots x_n]/\mathcal{I}(X)$.*

Proof. By definition, a regular function on X is defined by a polynomial. Therefore we have a surjective map $k[x_1, \dots x_n] \to \mathcal{O}(X)$ which is easily seen to be a ring homomorphism. The kernel consists of the polynomials which vanish on X, but this is precisely $\mathcal{I}(X)$. □

Corollary 1.2.3. *$\mathcal{O}(X)$ is a finitely generated integral domain,*

Given a morphism $f : X \to Y$ and a regular function $g \in \mathcal{O}(Y)$, $f^*g = g \circ f$ is regular function on Y. This yields a homomorphism $f^* : \mathcal{O}(Y) \to \mathcal{O}(X)$. If f is defined by (1.1), then f^* fits into a diagram

$$\begin{array}{ccc} k[y_1, \dots, y_m] & \xrightarrow{\tilde{F}} & k[x_1, \dots x_n] \\ \downarrow & & \downarrow \\ \mathcal{O}(Y) & \xrightarrow{f^*} & \mathcal{O}(X) \end{array}$$

where $\tilde{F}(y_i) = F_i(x_1, \dots, x_n)$.

Suppose that X is irreducible, then $\mathcal{I}(X)$ is prime. Therefrore $\mathcal{O}(X)$ is an integral domain. So we can form the field of fractions

$$K(X) = \left\{ \frac{f}{g} \mid f, g \in \mathcal{O}(X), g \neq 0 \right\}$$

An element of $K(X)$ is called a *rational function* on X, and $K(X)$ is referred to as the *function field*. Since $\mathcal{O}(X)$ is finitely generated as a k-algebra, $K(X)$ is obtained by adjoining a finite number of elements to k. By field theory, $K(X)$ is a finite extension of a totally transcendental extension $k(x_1, \dots, x_n)$. The elements x_i are far from unique, however the number n is well defined and called the transcendence degree of $K(X)$. We define the *dimension* of X to be the transcendence degree of $K(X)$. It follows immediately that $\dim \mathbb{A}^n_k = n$ as it should.

If $X \subset \mathbb{A}^n_k$ is an algebraic variety, a prime ideal J containing $\mathcal{I}(X)$ is called minimal if there is no smaller prime ideal J' containing $\mathcal{I}(X)$. If J is a minimal prime, $V(J) \subset X$ is called an *irreducible component* of X.

Let us call a finitely generated k-algebra *affine*.

Theorem 1.2.4 (Weak Duality)**.** *$\mathcal{O}$ induces a bijection between the set of isomorphism classes of affine varieties and the set of isomorphism classes of affine domains (finitely k-algebras which are integral domains).*

This is actually a corollary of a stronger and more useful statement.

Theorem 1.2.5 (Duality)**.** *$\mathcal{O}$ induces an anti-equivalence between the category of affine varieties and the category of affine domains and k-algebra homomorphisms. In more explicit terms, this means that*

(a) Every reduced affine algebra is isomorphic to $\mathcal{O}(X)$ for some X.

(b) *$Hom(X,Y) \cong Hom(\mathcal{O}(Y), \mathcal{O}(X))$ where the respective Hom's denote the set of morphisms/homomorphisms.*

Proof. An affine domain A is generated by a finite number of elements, say $a_1, \ldots, a_n$. Therefore we have a surjective homomorphism $k[x_1, \ldots x_n] \to A$ sending $x_i \mapsto a_i$. Then $A \cong k[x_1, \ldots x_n]/I$ where I denotes the kernel. Since A is a domain, I is prime and so $I = \mathcal{I}(X)$ where $X = V(I)$. Therefore $A \cong \mathcal{O}(X)$.

Let $X \subset \mathbb{A}^n_k, Y \subset \mathbb{A}^m_k$. As we have seen a morphism $f : X \to Y$ induces a homomorphism $f^* : \mathcal{O}(Y) \to \mathcal{O}(X)$. Conversely suppose that $F : \mathcal{O}(Y) \to \mathcal{O}(X)$ is a homomorphism. This can be realized as homomorphism $\tilde{F} : k[y_1, \ldots, y_m] \to k[x_1, \ldots x_n]$ which maps $I(Y)$ into $I(X)$. Set $F_i = \tilde{F}(y_i)$. Then this defines a morphism $f : X \to Y$ such that $f^* = F$. □

We are calling this "duality" because it can viewed as an analogue of Gelfand duality in point set topology/functional analysis.

Theorem 1.2.6 (Gelfand duality)**.** *There is an anti-equivalence between the categories of compact Hausdorff spaces and commutative Banach algebras.*

In one direction, a space X corresponds to the algebra of continuous complex valued functions $C(X)$. In the opposite direction, X, as a set, is given by the set of maximal ideals $\operatorname{Max} C(X)$. Returning to algebraic geometry, given a variety X, we can recover it from $\mathcal{O}(X)$ by the same strategy: Given $a \in X$, let

$$m_a = \{f \in \mathcal{O}(X) \mid f(a) = 0\}$$

Lemma 1.2.7. *This is a maximal ideal.*

Proof. Let $ev_a : \mathcal{O}(X) \to k$ be defined by $ev_a(f) = f(a)$. This is easily seen to be a surjective homomorphism. The kernel is precisely m_a. This shows that m_a is an ideal such that the quotient $\mathcal{O}(X)/m_a = k$. Since the quotient is a field, m_a is maximal. □

Theorem 1.2.8. *If k is algebraically closed then any maximal ideal of $\mathcal{O}(X)$ is of the form m_a for a unique $a \in X$.*

Proof. Let $X \subset \mathbb{A}^n_k$. Then we have surjective homomorphism $k[x_1, \ldots x_n] \to \mathcal{O}(X)$. Given a maximal ideal $m' \subset \mathcal{O}(X)$, the preimage $m \subset k[x_1, \ldots x_n]$ is again a maximal ideal. By the Nullstellensatz $V(m) \subset X$ is nonempty. Let $a \in V(m)$. Then $\mathcal{I}(a) \supseteq \mathcal{I}(V(m)) \supseteq m$. Since m is maximal, we must have equality $m = \mathcal{I}(a)$. Consequently $m' = m_a$. Finally observe that $m_a \neq m_b$ unless $a = b$. □

Therefore we have a natural bijection between X and the set of maximal ideals $\operatorname{Max} \mathcal{O}(X)$.

1.3 Examples/Exercises

Let k be an algebraically closed field.

Example 1.3.1. *Let us identify $\mathbb{A}_k^{n^2}$ with the set $Mat_{n\times n}(k)$ of $n \times n$ matrices. The set of singular matrices is algebraic since it is defined by the vanishing of the determinant det which is a polynomial.*

Example 1.3.2. *Then the set $SL_n(k) \subset \mathbb{A}^{n^2}$ of matrices with determinant 1 is algebraic since it's just $V(\det -1)$.*

The set of nonsingular matrices $GL_n(k)$ is not an algebraic subset of $Mat_{n\times n}(k)$. However, there is useful trick for identifying it with an algebraic subset of $\mathbb{A}^{n^2+1} = \mathbb{A}^{n^2} \times \mathbb{A}^1$.

Example 1.3.3. *The image of $GL_n(k)$ under the map $A \mapsto (A, 1/\det(A))$ identifies it with the algebraic set*

$$\{(A, a) \in \mathbb{A}^{n^2+1} \mid \det(A)a = 1\}$$

Example 1.3.4. *Identify $\mathbb{A}_k^{mn}$ with the set of $m \times n$ matrices $Mat_{m\times n}(k)$. Then the set of matrices of rank $\leq r$ is algebraic. This is because it is defined by the vanishing of the $(r+1) \times (r+1)$ minors, and these are polynomials in the entries. Notice that the set of matrices with rank equal r is not algebraic.*

Example 1.3.5. *The set of pairs $(A, v) \in Mat_{n\times n}(k) \times k^n$ such that v is an eigenvector of A is algebraic, since the condition is equivalent to $rank(A, v) \leq 2$.*

Example 1.3.6. *The map $p_i : \mathbb{A}_k^{n^2} \to \mathbb{A}_k^{n^2}$ which sends $A \mapsto A^i$ is regular. Therefore the subset $p_i^{-1}(0) \subseteq \mathbb{A}_k^{n^2}$ of matrices which are nilpotent of order i, i.e matrices A such that $A^i = 0$, are algebraic.*

Example 1.3.7. *To every matrix, A we can associate its characteristic polynomial $\det(tI - A)$. We thus get a morphism $ch : \mathbb{A}^{n^2} \to \mathbb{A}^n$ given by taking the coefficients of this polynomial other than the leading coefficient which is just one. The preimage $ch^{-1}(0)$ of matrices which are nilpotent (of unspeficied order) is algebraic.*

Let's look more closely at the last two examples when $n = 2$. Let A be 2×2 matrix over k. The Cayley-Hamilton theorem tells us that

$$A^2 - trace(A)A + det(A)I = 0$$

Therefore $det(A) = trace(A) = 0$ implies that A is nilpotent of order 2. Conversely, these vanish for a nilpotent matrix since it has zero eigenvalues. Let's try and understand this using the Nullstellensatz. Let

$$A = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$$

be a generic matrix over $R = k[x_1, \dots x_4]$. The polynomials $det(A), trace(A)$ generate an ideal $I \subset R$. The entries of A^2 generate another ideal J. We have just seen that $V(I) = V(J)$. Therefore $\sqrt{I} = \sqrt{J}$. But we can ask finer questions: does $I = J$? Which of these ideals is radical? We are going to use Grayson and Stillman's Macaulay2 program which is particularly convenient for algebraic geometry/commutative algebra. We will answer these in characteristic 0, however we need to work over a field where the elements and operations can be represented precisely on a machine. We will use the prime field $k = \mathbb{Q}$ even though we are interested in algebraically closed fields containing it. This is justified by the following:

Lemma 1.3.8. *Let $k \subset k'$ be a field extension. Suppose that $I \subset k[x_1, \dots x_n]$ is an ideal, and let $I' \subset k'[x_1, \dots x_n]$ be the ideal generated by I. Then $I' \cap k[x_1, \dots x_n] = I$ and $\sqrt{I'}$ is generated by $\sqrt{I}$.*

Proof. We prove this using tensor products (see Atiyah-MacDonald for a review). We have $k'[x_0, \dots x_n] = k' \otimes_{k_1} k[x_1, \dots x_n]$ as algebras. Furthermore $I' = k' \otimes_k I$ and the first statement follows easily.

Let $J = k' \otimes \sqrt{I}$. We certainly have $I' \subset J \subseteq \sqrt{I'}$, we just have to check that J is radical. This is equivalent to condition that the ring

$$k'[x_1, \dots x_n]/J \cong k' \otimes_k (k[x_1, \dots x_n]/\sqrt{I})$$

has no nonzero nilpotents. This is clear, since $(a \otimes f)^n = a^n \otimes f^n = 0$ forces a or f to be 0. $\square$

Beware that for some questions the field does matter. For example, the ideal $(x^2 - 2)$ is prime over $\mathbb{Q}$ but not over $\mathbb{C}$. Below is a transcript of a Macaulay 2 session which shows that $\sqrt{I} = \sqrt{J}$, that $J \subsetneq I$ and that I is radical. It shouldn't be too hard to understand what the commands mean. The semicolon is used to suppress output, the symbol = is used to assign values, and == for testing equality. We will revisit this example in the exercises.

```
i1 : R = QQ[x_1..x_4];

i2 : A = matrix{{x_1,x_2},{x_3,x_4}};

                2       2
o2 : Matrix R  <--- R

i3 :  D = det A;

i4 : T = trace A;

i5 : I =ideal {D,T}

o5 = ideal (- x x  + x x , x  + x )
               2 3    1 4   1    4
```

```
o5 : Ideal of R

i6 : J = ideal A^2

             2                                             2
o6 = ideal (x  + x x , x x  + x x , x x  + x x , x x  + x )
             1    2 3   1 3    3 4   1 2    2 4   2 3    4

o6 : Ideal of R

i7 : I == J

o7 = false

i8 : isSubset(J,I)

o8 = true

i9 : radical I == radical J

o9 = true

i10 : radical I == I

o10 = true
```

Before doing the next example, let us review resultants. Given two polynomials

$$f = a_n x^n + \dots a_0$$

and

$$g = b_m x^m + \dots b_0$$

Suppose, we wanted to test whether they had a common zero, say α. Then multiplying $f(\alpha) = g(\alpha) = 0$ by powers of α yields

$$\begin{array}{ccccccc}
 & & a_n\alpha^n + & a_{n-1}\alpha^{n-1} + \dots & a_0 & = & 0 \\
 & & \dots & & & & \\
a_n\alpha^{n+m} + & a_{n-1}\alpha^{n+m-1} + \dots & a_0\alpha^m & & & = & 0 \\
 & & & b_m\alpha^m + \dots & b_0 & = & 0 \\
 & & \dots & & & & \\
b_m\alpha^{n+m} + \dots & & & & & = & 0
\end{array}$$

We can treat this as a matrix equation,

$$\begin{pmatrix} 0 & \cdots & a_n & \cdots & a_0 \\ & & \cdots & & \\ a_n & \cdots & a_0 & \cdots & 0 \\ 0 & \cdots & b_m & \cdots & b_0 \\ & & \cdots & & \end{pmatrix} \begin{pmatrix} \alpha^{n+m} \\ \alpha^{n+m-1} \\ \vdots \\ \alpha \\ 1 \end{pmatrix} = 0$$

For a solution to exist, we would need the determinant of the coefficient matrix, called the resultant of f and g, to be zero. The converse, is also true (for $k = \bar{k}$) and can be found in most standard algebra texts. Thus:

Example 1.3.9. *Identify the set of pairs (f, g) with $\mathbb{A}^{(n+1)+(m+1)}$. The set of pairs with common zeros is algebraic.*

We can use this to test whether a monic polynomial (i.e. a polynomial with leading coefficient 1) f has repeated root, by computing the resultant of f and its derivative f'. This called the discriminant of f. Alternatively, if we write $f(x) = \prod(x - r_i)$, the discriminant $disc(f) = \prod_{i<j}(r_i - r_j)^2$. This can be written as a polynomial in the coefficients of f by the theorem on elementary symmetric functions.

Example 1.3.10. *The set of monic polynomials with repeated roots is an algebraic subset of $\mathbb{A}^n$.*

Example 1.3.11. *The set of matrices in $\mathbb{A}^{n^2}$ with repeated eigenvalues is an algebraic set. More explicitly it is the zero set of the discriminant of the characteristic polynomial.*

Exercise 1.3.12.

1. *Identify $\mathbb{A}^6 = (\mathbb{A}^2)^3$ with the set of triples of points in the plane. Which of the following is algebraic and why:*
   (a) *The set of triples of distinct points.*
   (b) *The set of triples (p_1, p_2, p_3) with $p_3 = p_1 + p_2$.*
   (c) *The set of triples of colinear points.*
2. *Recall that a square matrix is unipotent if 1 is its only eigenvalue. Prove that the set of unipotent matrices in $\mathbb{A}^{n^2}$ is algebraic.*
3. *Give a* proof *that the ideal $I = (trace(A), det(A))$, where A is the generic matrix over $k[x_1, x_2, x_3, x_4]$, is not just radical but prime for any k.*
4. *Prove that topological space X is irreducible if and only if every nonempty open subset is dense.*
5. *Recall that a matrix is diagonalizable if it is similar to a diagonal matrix. Prove that the set of diagonalizable matrices in $\mathbb{A}_k^{n^2}$ is dense in the Zariski topology but not open.*

6. *Let* $X \subseteq \mathbb{A}^n_k, Y \subseteq \mathbb{A}^m_k$*, show that their Cartesian product* $X \times Y \subset \mathbb{A}^{n+m}_k$ *is algebraic. However, show that the Zariski topology on* $\mathbb{A}^{n+m}_k$ *is* **not** *the product topology.*

7. (a) *Show that the composition of two morphisms (resp. isomorphisms) of affine varieties is again a morphism (resp isomorphism).*

   (b) *Show that an isomorphism is a bijection, and that it induces an isomorphism of coordinate rings.*

   (c) *However, show that a bijective morphism is not necessarily an isomorphism by considering the morphism* $\mathbb{A}^1 \to V(y^2 - x^3)$ *sending* $t \mapsto (t^2, t^3)$.

8. *Assume that the characteristic of* k *is different from* 2*. Prove that* $X = V(xz - y^2)$ *is irreducible. Consider the regular map* $f : \mathbb{A}^2_k \to X$ *given by* $f(u, v) = (u^2, uv, v^2)$*. Is it bijective? Is it an isomorphism? Explain.*

9. *Denote the elements of* $\mathbb{A}^2_k$ *by column vectors. Show that the following transformations* $T : \mathbb{A}^2_k \to \mathbb{A}^2_k$ *are automorphisms and that the two classes (a) and (b) form subgroups of the automorphism group.*

   (a) $Tv = Av + b$ *where* $A \in GL_2(k)$, $b \in \mathbb{A}^2$.

   (b)
$$T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y + f(x) \end{pmatrix} D$$
   where $f(x) \in k[x]$

10. *Let* $X \subseteq \mathbb{A}^n_k$ *be an affine variety. Give it the induced topology (a subset of* X *is closed if it is closed in* $\mathbb{A}^n_k$*). Given an ideal* $I \subset \mathcal{O}(X)$*, define* $V(I) = \{a \in X \mid f(a) = 0\}$.

    (a) *Show closed sets are of this form.*

    (b) *Extend the Nullstellensatz to give a bijection between closed subsets of* X *and radical ideals in* $\mathcal{O}(X)$.