Chapter 11

Complex algebraic curves

11.1 Topology of curves

Let us assume $k = \mathbb{C}$ throughout this chapter. We give $\mathbb{P}^n_{\mathbb{C}}$ the classical topology induced from the Euclidean metric. This is compact Hausdorff. Given a nonsingular projective curve $X \subset \mathbb{P}^n_{\mathbb{C}}$ can be given the induced topology which is what now use. It can shown to be independent of the embedding. We define the sheaves $\mathcal{O}_X^{an}$ and C_X^∞ of holomorphic and (complex valued) C^∞ functions as follows. $f \in \mathcal{O}_X(U)$ if locally f extends to a holomorphic function on $\mathbb{P}^n_{\mathbb{C}}$. The C^∞ case is similar. When equipped with these sheaves, X be comes a compact one dimensional complex manifold or two dimensional C^∞ manifold. For historical reasons, one dimensional complex manifolds are usually called Riemann surfaces. The topological classification of these is well understood: it is given by g holed torus for some unique $g \in \mathbb{N}$ usually called the genus, but which we will call the classsical genus. For instance $\mathbb{P}^1_{\mathbb{C}}$ is a sphere, so it has genus is zero. Pictured below is a genus 2 surface.

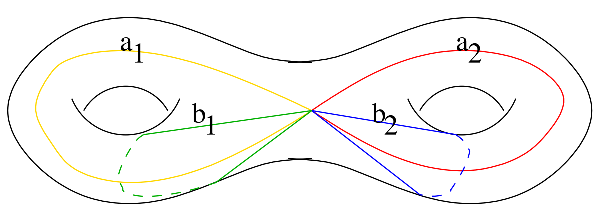

Figure 11.1: Genus 2 surface

To give a more rigorous definition of classical genus, we need a bit of algebraic topology. Given a topological space X, the elements of first homology $H_1(X, \mathbb{Z})$ are equivalence classes of oriented loops, where two loops are equivalent if their difference is the boundary of something. In the example pictured above, a_1, a_2, b_1, b_2 gives a basis; two per "hole". In general, we have

Theorem 11.1.1. *If X smooth projective curve over $\mathbb{C}$, $H_1(X, \mathbb{Z})$ is a free abelian group of rank equal to 2 times the genus.*

There are really two assertions. First that $\operatorname{rank} H_1(X, \mathbb{Z})$ is even, and in particular finite dimensional. Second that the rank equals $2g$. Assuming the first part, which we will prove, define the classical genus $c = \frac{1}{2} \operatorname{rank} H_1(X, \mathbb{Z})$. Then the claim is $g = c$. We will prove a slight weaker statement, after the basic tools are introduced.

11.2 de Rham cohomology

Rather than use homology, we will work with the dual notion of de Rham cohomology, where we can develop the basic properties *ab initio*. Given a C^∞ (real) 2-manifold X, let $\mathcal{E}(X)$ denote the space of complex valued C^∞ 1-forms. An element of this ω is locally given by

$$\omega = f(x, y)dx + g(x, y)dy$$

with C^∞ coefficients. We say that ω is exact if there exists $h \in C^\infty(X)$ such that $\omega = dh$, and closed if

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$$

holds locally. On a disk, these notions are equivalent.

Proposition 11.2.1 (Poincaré's lemma)**.** *A 1-form on a disk is exact if and only if it is closed.*

This is easy to prove using integration. And this is something you do (perhaps implicitly) in a differential equations class. In general, exact always implies closed, but not conversely. The failure is measured by the first de Rham cohomology

$$H^1(X, \mathbb{C}) = \frac{\{\text{closed } \mathbb{C}\text{-valued 1-forms}\}}{\{\text{exact } \mathbb{C}\text{-valued 1-forms}\}}$$

$H^1(X, \mathbb{R})$ can be defined the same way. A special case of de Rham's theorem says that

Theorem 11.2.2 (de Rham)**.**

$$H^1(X, \mathbb{C}) \cong Hom(H_1(X, \mathbb{Z}), \mathbb{C})$$

$$H^1(X, \mathbb{R}) \cong Hom(H_1(X, \mathbb{Z}), \mathbb{R})$$

where the map sends ω to the line integral $\gamma \mapsto \int_\gamma \omega$.

Corollary 11.2.3. $\dim_{\mathbb{C}} H^1(X, \mathbb{C}) = \dim_{\mathbb{R}} H^1(X, \mathbb{R}) = \operatorname{rank} H_1(X, \mathbb{Z})$

Since we didn't really define H_1 very carefully, we are not in position to prove this. But it least tells us what information $H^1(X, \mathbb{C})$ contains.

We want to prove that $\dim H^1(X) < \infty$ when X is compact. It is convenient to also define the 0th cohomology

$$H^0(X, \mathbb{C}) = \{f \mid df = 0\}$$

Of course, these are just constant functions, when X is connected.

Theorem 11.2.4 (Mayer-Vietoris)**.** *If $X = U \cup V$ is union of open sets, there is a long exact sequence*

$$0 \to H^0(X, \mathbb{C}) \to H^0(U, \mathbb{C}) \oplus H^0(V, \mathbb{C}) \to H^0(U \cap V, \mathbb{C}) \to$$

$$H^1(X, \mathbb{C}) \to H^1(U, \mathbb{C}) \oplus H^1(V, \mathbb{C}) \to H^1(U \cap V, \mathbb{C})$$

Proof. We have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & C^\infty(X) \arrow[r, "s"] \arrow[d, "d"] & C^\infty(U) \oplus C^\infty(V) \arrow[r, "d"] \arrow[d, "d"] & C^\infty(U \cap V) \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & \mathcal{E}_{cl}(X) \arrow[r, "s"] & \mathcal{E}_{cl}(U) \oplus \mathcal{E}_{cl}(V) \arrow[r, "d"] & \mathcal{E}_{cl}(U \cap V) \arrow[r] & 0
\end{tikzcd}$$

where s and d are the sum and difference of restrictions, and $\mathcal{E}_{cl}$ is the space of closed 1-forms. The theorem now follows from the snake lemma. □

Corollary 11.2.5. *If X is compact, then* $\dim H^1(X, \mathbb{C}) < \infty$.

Proof. By compactness, X has a finite open cover by disks. We prove the corollary for any X which admits such a cover $\{U_1, \ldots, U_n\}$ by induction. By Mayer-Vietoris,

$$H^0((U_1 \cup \ldots U_{n-1}) \cap U_n) \to H^1(X) \to H^1(U_1 \cup \ldots U_{n-1}) \oplus H^1(U_n)$$

The dimension of the left side is the number of connected components of $(U_1 \cup \ldots U_{n-1}) \cap U_n$ and the dimension of the right is finite by induction.

□

11.3 The genus is bounded by classical genus

We will prove the following

Theorem 11.3.1. *If X smooth projective curve over* $\mathbb{C}$, $\dim H^1(X, \mathbb{C}) = 2c$ *for some c, and the genus $g \leq c$.*

In fact, $g = c$, but the proof of this is harder. To prove evenness, we need a standard result from linear algebra.

Theorem 11.3.2. *If a finite dimensional real or complex vector space V carries a nondegenerate alternating form $\langle , \rangle$, then the dimension must be even. Furthermore we can find a basis such that the form is represented by*

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

If subspace $W \subseteq V$ satisfies $\forall w, w' \in W, \langle w, w' \rangle = 0$ (W is called isotropic) then $\dim W \leq \frac{1}{2}V$.

Before constructing the alternating form, let us quickly recall some facts from calculus. A 2-form on a region $D \subset \mathbb{R}^2$ is integrated by

$$\int_D (-) dx \wedge dy = \iint_D (-) dx dy$$

$$\int_D (-) dy \wedge dx = -\iint_D (-) dx dy$$

This asymmetry has to do with the orientation of the plane. An orientation is a choice of nowhere zero real 2-form (or n-form on an n-manifold), which tells us which ordering is positive. Also the rules of exterior algebra imply

$$(f dx + g dy) \wedge (h dx + k dy) = (fk - gh) dx \wedge dy$$

Finally Stokes' theorem says

$$\int_D df = \int_{\partial D} f$$

when D has a smooth boundary ∂D orientated counterclockwise. To integrate on a general two manifold X, we need to choose and fix an orientation: a Riemann surface has a preferred orientation, which we use. We define a bilinear form on $\mathcal{E}(X)$ by

$$\langle \alpha, \beta \rangle = \int_X \alpha \wedge \beta$$

Lemma 11.3.3. *This gives an alternating form on $H^1(X, \mathbb{C})$*

Proof. It is clearly alternating, since

$$\langle \alpha, \beta \rangle = -\langle \beta, \alpha \rangle$$

We have

$$\langle df, \beta \rangle = \int_X d(f\beta) = 0$$

by Stokes, since X has no boundary. Therefore this gives a well defined pairing on cohomology. □

Theorem 11.3.4 (Poincaré duality)**.** *This is a nondegenerate alternating pairing on $H^1(X, \mathbb{C})$, i.e. it is represented by a nonsingular matrix.*

Proof. Note that $H^1(X, \mathbb{C}) = H^1(X, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$, and the pairings are compatible. So we may as well work with real coefficients, which is a bit simpler. By linear algebra, the nondegeneracy amounts to the following: if $\omega \neq 0$, then we need to find ω' such that
$$\langle \omega, \omega' \rangle \neq 0$$

Given a holomorphic local coordinate z, let x and y be the real and imaginary parts. Define the Hodge star operator by

$$*dx = dy,\ *dy = -dx$$

This can viewed as multiplication by i in the real cotangent planes, so it is independent of coordinates. One can see that for a real 1-form

$$\omega = f dx + g dy$$

$$\omega \wedge *\omega = (f^2 + g^2) dx \wedge dy$$

Therefore if $\omega \neq 0$, then

$$\langle \omega, *\omega \rangle = \int_X \omega \wedge *\omega > 0$$

□

This gives part of what were after.

Corollary 11.3.5. $\dim H^1(X, \mathbb{C})$ *is even.*

To finish what we set out to prove, we need

Theorem 11.3.6. $g \leq c$

First, we prove an analogue of the fact that regular functions on a projective variety are constant.

Lemma 11.3.7. *A holomorphic function on X is constant.*

Proof. Assume that h is nonconstant. By compactness of X, $|h|$ will attain a maximum somewhere, say at $p \in X$. Since h is nonconstant and holomorphic, $h - h(p)$ will have isolated zeros. Therefore h is nonconstant in any disk D containing centered at p. On the other hand, the maximum modulus principle, implies that h is constant because $|h|$ has a maximum at an interior point. So we have a contradiction. □

Proof of theorem. Let $V = \Omega_X(X)$. Any element $\omega \in V$ is holomorphic, i.e. in a holomorphic local coordinate $\omega = f(z)dz$, where $f(z)$ is holomorphic. Setting $u = Re(f)$ and $v = Im(f)$, we have

$$\omega = (u + iv)dx + (-v + iu)dy$$

Then the Cauchy-Riemann equations

$$u_x = v_y,\, u_y = -v_x$$

imply that ω is closed. This gives a map

$$V \to H^1(X, \mathbb{C})$$

We claim that the above map is injective. Suppose that $\omega \in V$ is in the kernel. Then $\omega = dh$ for some C^∞ function h on X. If we set $s = Re(h), t = Im(h)$, then

$$s_x = u = t_y,\, t_x = v = -s_y$$

Therefore h is holomorphic. Since h is globally defined, it is constant by previous lemma. Therefore we $\omega = dh = 0$. So the claim is proved. This already implies $g \leq 2c$. To get the stronger inequality, it is enough to show that V is isotropic. Suppose that $\omega, \eta \in V$ then locally $\omega = f dz, \eta = g dz$. Therefore

$$\omega \wedge \eta = fg dz \wedge dz = 0$$

so that

$$\langle \omega, \eta \rangle = \int_X \omega \wedge \eta = 0$$

Therefore V is isotropic.

□

11.4 Exercises

Exercise 11.4.1.

1. *Using the Poincaré lemma and Mayer-Vietoris, show that* $H^1(S^2, \mathbb{C}) = 0$, *where* S^2 *is the unit sphere in* $\mathbb{R}^3$ *or you take it to be* $\mathbb{P}^1_{\mathbb{C}}$.

2. *Let* $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ *be the standard torus. Show that* $H^1(T, \mathbb{C}) = \mathbb{C}^2$ *with generators corresponding to* dx, dy*, where* x, y *be the standard coordinates on the plane.*

3. *Let* $E = \mathbb{C}/\mathbb{Z} + \mathbb{Z}i$ *viewed as a Riemann surface. Show the space of holomorphic* 1*-forms is one dimensional and spanned by* dz.