Chapter 5

Introduction to Schemes

5.1 Spec of a ring

So far we have been focusing on quasiprojective varieties over an algebraically closed field. But this is often not good enough for several reasons:

- People who work in commutative algebra or number theory use a lot of algebraic geometry, but they usually don't work over algebraically closed fields. Even in classical algebraic geometry, it is natural to work over function fields.
- In order for something to be quasiprojective, it has to be given as a subset of projective space, but this is often complicated and besides the point. This came up in the construction of products for example.
- Even in classical geometry, the language of quasiprojective varieties is not completely adequate. For example consider the family of conics $ty - x^2 = 0$, as $t \to 0$ it more natural to view this degenerating to a "double line".

The solution to all of these problems, due to Grothendieck, is to consider something more general called a scheme.[1] The definition is pretty involved. So let's start by describing the basic building blocks called affine schemes.

Recall that given an affine variety X over algebraically closed field, we can associate the affine domain $R = \mathcal{O}(X)$. We can recover X from R. From the Nullstellensatz

$$a \mapsto m_a = \{f \in R \mid f(a) = 0\}$$

gives a bijection between the set X and the space of maximal ideals $\operatorname{Max} R$. A regular function $F : X \to Y$ to another affine variety determines a homomor-

[1] Of course in ordinary English it has an entirely different meaning, but I believe the Greek root means "figure" or "form". The original source is Grothendieck and Dieudonné's EGA=Éléments de Géometrie Algébrique, which like Euclid's Elements is actually several books. Nowadays most people use Hartshorne's Algebraic Geometry as an introduction to this topic.

phism $F^* : S \to R$, where $S = \mathcal{O}(Y)$. The following is pretty straightforward with the help of the Nullstellensatz

Lemma 5.1.1. *Given a maximal ideal $m_a \in \operatorname{Max} R$, $f^{-1}m_a$ is a maximal ideal of S which is in fact $m_{f(a)}$.*

Grothendieck's idea was that instead of restricting to some special class of rings, start with an arbitrary commutative ring R and build a corresponding space. However, we have to make some adjustments. In this generality, maximal ideals don't work so well. If $F : S \to R$ is a homomorphism of commutative rings, then it is not usually true that F^{-1} takes maximal ideals to maximal ideals. What is true and easy is to see is that F^{-1} takes maximal ideals to prime ideals, and more generally prime ideals to prime ideals. With that in mind, we define the (prime) spectrum $\operatorname{Spec} R$ to be the set of prime ideals. Then, we observe:

Lemma 5.1.2. Spec *is a contravariant functor from commutative rings to sets.*

Next we introduce a topology, also called the Zariski topology. Given an ideal $I \subset R$, let us define $V_{new}(I)$ as

$$V_{new}(I) = \{p \in \operatorname{Spec} R \mid I \subseteq p\}$$

Also for $f \in R$, define

$$D_{new}(f) = \{p \mid f \notin p\}$$

At some point, we will drop the subscript "new".

Lemma 5.1.3. $\operatorname{Spec} R$ *carries a unique topology, again called the Zariski topology, whose closed sets are the $V_{new}(I)$. The $D_{new}(f)$'s give a basis for the open sets.*

A proof can be found in many places, such as Hartshorne. Let see what this looks like in the simple example of $R = k[x]$, where k is an algebraically closed field. Using the fact that R is a PID, we see easily that

$$\operatorname{Spec} R = \operatorname{Max} R \cup \{(0)\}$$

The first set is in bijection with $\mathbb{A}^1_k$, but get a new point $\eta = (0)$, called the generic point. We can distinguish these different points topologically.

Lemma 5.1.4. *Let R be a commutative ring. $p \in \operatorname{Spec} R$ is maximal if and only if $\overline{\{p\}} = \{p\}$. If R is an integral domain, then $\overline{\{\eta\}} = \operatorname{Spec} R$, where $\eta = (0)$.*

Proof. This will be an exercise. □

In general, if $R = \mathcal{O}(X)$, where X is affine, $\operatorname{Spec} R$ will contain many new points corresponding to irreducible subsets of X. However, we do get an inclusion $X \hookrightarrow \operatorname{Spec} R$ sending $a \mapsto m_a \in \operatorname{Max} R$. Even though $\operatorname{Spec} R$ is bigger than X, the partially ordered set of opens $Open(X)$ and $Open(\operatorname{Spec} R)$ are isomorphic by the following:

Lemma 5.1.5. *The injection $X \hookrightarrow \operatorname{Spec} R$ is continuous. Furthermore, $U \mapsto U \cap X$ is an order preserving bijection between the sets $Open(X)$ and $Open(\operatorname{Spec} R)$.*

Proof. From the definitions, we can see that $D(f) = D_{new}(f) \cap X$. So opens, which are unions of $D(f)$'s on $\operatorname{Spec} R$ restrict to opens in X. Given $V = \bigcup_i D(f_i)$, $U = \bigcup_i D_{new}(f_i)$ gives an open such that $V = U \cap X$. Alternately, $p \in U$ if and only if it corresponds to an irreducible subset of X meeting V. This shows U is uniquely determined from V.

□

5.2 Affine schemes

The topological space $\operatorname{Spec} R$ is fairly weak invariant. It cannot distinguish one field from another. What we are missing is the analogue of regular function. What should this be? The answer is that an element of R should play the role of a function on $\operatorname{Spec} R$. We also have to say what a function is on a subset. Before we can do this, we need to understand sheaves in a more serious way.

A *presheaf* of sets, groups rings... on a topological space X consists of an assignment of $\mathcal{F}(U)$ of a set, group... to each open $U \subset X$ and a map or homorphism $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ (called restriction) for each pair $V \subseteq U$, such that

1. $\rho_{UU} = id$
2. If $W \subset V \subset U$, then $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$.

In practice, we write f_V instead of $\rho_{UV}(f)$. Elements of $\mathcal{F}(U)$ are called sections, and global sections when $U = X$. Another way to say what a presheaf is is to regard $Open(X)$ as a category, where objects are open sets and morphisms are inclusions $U \subseteq V$. Then a presheaf is the same things as contravariant functor from $Open(X)$ to Sets, or whatever. There are many examples which have nothing to do with algebraic geometry.

A presheaf $\mathcal{F}$ on X is called a *sheaf*, if for any open set U and open cover $U = \bigcup U_i$. Given a collection $f_i \in \mathcal{F}(U_i)$ such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$, there exists a unique $f \in \mathcal{F}(U)$ with $f|_{U_i} = f_i$. In general, a presheaf of functions is a sheaf if the defining conditions are *local*. E.g. the presheaf of continuous functions on $\mathbb{R}^n$, $\mathbb{C}^\infty$ functions on $\mathbb{R}^n$, holomorphic functions on $\mathbb{C}$ are sheaves. The presheaf of bounded functions on $\mathbb{R}^n$ is not (exercise).

Theorem 5.2.1. *There exists a unique sheaf of rings $\mathcal{O}_X$ on $X = \operatorname{Spec} R$ such that*

$$\mathcal{O}_X(D_{new}(f)) = R\left[\frac{1}{f}\right]$$

with restrictions of sections under $D_{new}(gf) \subset D_{new}(f)$ given by natural maps

$$R\left[\frac{1}{f}\right] \to R\left[\frac{1}{gf}\right]$$

A detailed construction can found in Hartshorne. However, in fact everything can be read off from the above description. Suppose $U = D_{new}(f) \cup D_{new}(g)$ and we are wondering what $\mathcal{O}_X(U)$ looks like. The sheaf property tells us that it would be given by a pair $(r_1, r_2) \in R[1/f] \times R[1/g]$ with r_i having the same image in $R[1/fg]$.

Definition 5.2.2. *A ringed space is a pair consisting of a topological space and a sheaf of commutative rings on it. The affine scheme associated to R is the ringed space $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$. This is usually also denoted by $\operatorname{Spec} R$.*

Lemma 5.2.3. *The scheme $\operatorname{Spec} R$ determines R*

Proof. $R = \mathcal{O}_{\operatorname{Spec} R}(\operatorname{Spec} R)$. □

Let X be an affine variety over an algebraically closed field with $R = \mathcal{O}(X)$. We saw earlier that we have an inclusion $X \subset \operatorname{Spec} R$, such that open sets correspond. Now let's compare functions.

Lemma 5.2.4. *A section of $\mathcal{O}_{\operatorname{Spec} R}(U)$ restricts to a regular function on $X \cap U$. Any regular on $X \cap U$ extends to a section of $\mathcal{O}_{\operatorname{Spec} R}(U)$.*

Proof. By the sheaf property, we can work with basic opens. Then this comes down to the isomorphisms

$$\mathcal{O}_{\operatorname{Spec} R}(D_{new}(f)) = R[1/f] = \mathcal{O}_X(D(f))$$

□

The import of lemmas 5.1.5 and 5.2.4 is that there is no essential difference between X and $\operatorname{Spec} R$. From here on, we will just write V, D instead of V_{new}, D_{new}. It will be clear from context, which sense we mean.

Let us redefine affine space as $\mathbb{A}^n_R = \operatorname{Spec} R[x_1, \ldots, x_n]$. Given an ideal $I \subset R[x_1, \ldots, x_n]$, we can identify the space underlying $\operatorname{Spec} R[x_1, \ldots, x_n]/I$ with closed subset $V(I) \subseteq \mathbb{A}^n_R$ by the exercises. We call this a closed subscheme of affine space. For example the double line mentioned earlier can now be rigorously defined as the closed subscheme $\operatorname{Spec} k[x,y]/(x^2)$ of $\mathbb{A}^2_k$. As sets $V((x^2)) = V(x)$ but the schemes are different.

5.3 Exercises

Exercise 5.3.1.

1. *Prove lemma 5.1.1*
2. *Given an ideal $I \subset R$, show that the map $\operatorname{Spec} R/I \to \operatorname{Spec} R$ is injective, and the image is precisely $V(I)$.*
3. *The nilradical $\sqrt{0}$ of a ring R is the ideal of nilpotent elements. Show that $\operatorname{Spec} R/I = \operatorname{Spec} R$ if and only if $I \subseteq \sqrt{0}$.*

4. *Show that* $\operatorname{Spec} R$ *is noetherian if* R *is noetherian. Is the converse true? (Hint: you can use the previous problem.)*

5. *Prove lemma 5.1.4*

6. *Show that the presheaf of bounded functions on* $\mathbb{R}$ *is not a sheaf.*