Chapter 7

Schemes III

7.1 Functor of points

Here is another way to understand what a scheme is[1]. Given a scheme X, and a commutative ring R, the set of *R-valued points*

$$X(R) = Hom_{Schemes}(\operatorname{Spec} R, X)$$

This is in fact a functor from the category of commutative rings to sets. To get a sense of what this does, let $X = \operatorname{Spec} k[x_1, \dots, x_n]/(f_1, \dots, f_m)$ where k is a field. Suppose L is also a field. Then we know that

$$X(L) = Hom_{Rings}(k[x_1, \dots x_n]/(f_i), L)$$

To give such a homomorphism is to give an inclusion $k \subseteq L$ and an assigment $x_i \mapsto a_i \in L$ such that $f_i(a_1, a_2, \dots) = 0$. Therefore

Lemma 7.1.1. $X(L) = V(f_1, f_2, \dots) \subseteq \mathbb{A}^n_L$

Thus points are points in more or less the original sense. However, we have extended it allow us to work in bigger fields. Also we don't have to restrict to fields. Consider the the double line $X = \operatorname{Spec} k[x, y]/(x^2) \subset \mathbb{A}^2_k$ versus the ordinary line $Y = \operatorname{Spec} k[x, y]/(x)$. For any field $L \supseteq k$, $X(L) = Y(L)$, so we can't detect any difference this way. However, letting $R = k[\epsilon]/(\epsilon^2)$ gives

$$(\epsilon, a) \in X(R) - Y(R), \forall a \in k$$

So we can "see" the difference using these more general rings.

If X is nonaffine, we can choose an affine open cover $\{U_i\}$. Then

$$X(L) = \bigcup U_i(L)$$

[1]Unfortunately, Hartshorne omits this important topic, but there are plenty of other sources: EGA I, Demazure-Gabriel's "Intro to AG", Eisenbud-Harris' "Geometry of Schemes", Mumford's "Curves on an algebraic surface"...

Let us see how this works for $X = \mathbb{P}^n_k = \operatorname{Proj} k[x_0, \ldots x_n]$. We saw that there is an open cover $U_i \cong \operatorname{Spec} k[x_0/x_i, \ldots]$. Thus a point of $\mathbb{P}^n_k(L)$ is a point of one of these affine spaces $U_i(L)$. In the general, a given point would lie in several U_i's. We can see that $p \in U_i(L)$ and $q \in U_j(L)$ represent the same point if they are projectively the same in the sense that $[p] = [q]$. Therefore

Lemma 7.1.2. *$\mathbb{P}^n_k(L) = \mathbb{P}^n_L$, where the right hand side has the original meaning.*

Theorem 7.1.3. *A scheme is determined by its functor of points.*

We won't prove it, but we indicate the main steps. Given a scheme X, define the generalized functor of points by

$$S \mapsto Hom_{Schemes}(S, X)$$

The proof reduces to checking two statements

1. The generalized functor of points is determined by the functor of points.

2. A scheme is determined by its generalized functor of points.

1. comes down to the fact the generalized functor applied to $\operatorname{Spec} R$ is the original $X(R)$, a general scheme S is covered by affine schemes, and $X(-)$ is determined what happens on a covering. To make the last statement precise, we observe that

Lemma 7.1.4. *$X(-)$ is a sheaf.*

Sketch. Given $F = (f, f^{\#}) :\in X(S)$ and an open cover $\{U_i\}$. We get restrictions $F|_{U_i} : U_i \to X$ given by $f|_{U_i}$ and $\mathcal{O}_X(U) \to \mathcal{O}_S(f^{-1}U \cap U_i)$. It is easy to see that F is determined uniquely by the $F|_{U_i}$. In the other direction, given a collection of morphisms $U_i \to X$ which agree on intersections, we have to build an $F : S \to X$ such that $F_i = F|_{U_i}$. For f this is easy, because continuous functions can be patched. To define $f^{\#} : \mathcal{O}_X(U) \to \mathcal{O}_S(f^{-1}U)$ send g to the unique section γ such that

$$\gamma|_{f^{-1}U \cap U_i} = f_i^{\#}(g)$$

□

2. follows from a basic fact from category theory

Lemma 7.1.5 (Yoneda's lemma)**.** *If X and Y are objects of a category C such that there is a natural isomorphism $Hom_C(-, X) \cong Hom_C(-, Y)$, then $X \cong Y$.*

Sketch. One has bijections $Hom_C(X, X) \cong Hom_C(X, Y)$ and $Hom_C(Y, Y) \cong Hom_C(Y, X)$, under which the identities map to $f \in Hom_C(X, Y)$ and $g \in Hom_C(Y, X)$. Naturality can used to show that f and g are inverses. Therefore $X \cong Y$. □

7.2 Infinitessimals

Let k be a field. Suppose

$$R = k[x_1, \dots, x_n]/(f_1, \dots, f_m)$$

and $X = \operatorname{Spec} R$. Given $p \in X(k)$, imitating what we do in calculus, define the tangent space

$$T_pX = \{c \in k^n \mid \sum \frac{\partial f}{\partial x_i}|_p(c_i) = 0\} \tag{7.1}$$

Observe that this definition *depends on* the equations f_i, and not just X as scheme. We want to give an alternative description which is more intrinsic. The key is the ring $D = k[\epsilon]/(\epsilon^2)$, which came up earlier. For some reason, it is called the ring of dual numbers. We observe a couple of features of this ring. Firstly, D has exactly one prime ideal $m = (\epsilon)$. Given a polynomial $f(x_1, x_2, \dots) \in k[x_1, \dots x_n]$, and $a = (a_1, \dots) \in k^n$, we have a Taylor expansion

$$f(x_1, \dots) = f(a) + \sum \frac{\partial f}{\partial x_i}|_a(x_i - a_i) + \dots$$

where the ... refers to sum of quadratic and higher order homogeneous polynomials in $x_i - a_i$. If we substitute $a = b + c\epsilon \in k^n$ then all the higher terms will cancel. Therefore

Lemma 7.2.1.

$$f(a) = f(b) + \sum \frac{\partial f}{\partial x_i}|_a(c_i)\epsilon$$

A point $a \in X(D)$ is given by assigning $a_i = b_i + c_i\epsilon \in D$ such that $f(a) = 0$. From the previous lemma, we see that this equivalent to $b \in X(k)$ and $c \in T_bX$. Therefore

Lemma 7.2.2. *There is a bijection between $X(D)$ and pairs $b \in X(k)$ and $c \in T_bX$.*

Corollary 7.2.3. *There is a bijection $T_bX = \pi^{-1}(b)$, where $\pi : X(D) \to X(k)$ is the natural map.*

The last result shows that the set T_bX doesn't depend on the equations. But we still don't see the vector space structure. This can be obtained as follows. Let $b \in X(k)$, and let $m = m_b$ be the maximal ideal of regular function vanishing at b. Set $S = R_m$. This is a local ring with maximal ideal we also call m, and residue field k.

Lemma 7.2.4. $T_bX \cong Hom_k(m/m^2, k)$

Sketch. We can identify T_bX with the set of homomorphisms from $R \to D$ which send m to the maximal ideal of D. Such a homomorphism extend uniquely to local homomorphisms $f : S \to D$, and conversely. Since k is also a subfield of S, it splits into a sum $S = k \oplus m$. f is determined by its restriction $f|_m \in Hom_k(m/m^2, k)$. This defines a map $T_bX \to Hom_k(m/m^2, k)$, which can be seen to be an isomorphism. □

It is not hard to see that the vector space structure on the right agrees with the vector space structure on T_bX from (7.1). Given a local ring S with maximal ideal m and residue field k, we define its (Zariski) tangent space by

$$TS = Hom_k(m/m^2, k)$$

Observe that m/m^2 is always k-vector space, so this formula makes sense in general. Suppose that S is noetherian, then this space is finite dimensional. From commutative algebra, we get an inequality

Theorem 7.2.5. *If S is noetherian*

$$\dim TS \geq \dim S,$$

where the dimension on the left is the vector space dimension, and on the right it is the Krull dimension.

Rather than taking this on faith, we should understand why this holds, at least in the case when S is the local ring of a subscheme $X \subset \mathbb{A}^n_k$ as a above. From the inclusion, we get $\dim X \leq \dim \mathbb{A}^n = n$. If it so happens that $T_bX = k^n$, then we are done. But this usually doesn't happen. So instead try to replace $\mathbb{A}^n$ by $\mathbb{A}^{\dim T_bX}$ in such a way that it still contains X or enough of it, to compute the dimension. To make sense of this, we switch to the algebraic viewpoint. By Nakayama's lemma, $d = \dim T_bX$ can be identified with the minimum number of generators for the ideal m. Choose generators $\bar{y}_1, \ldots, \bar{y}_d \in m$, and define the k-algebra map $k[y_1, \ldots, y_d] \to S$ by sending $y_i \mapsto \bar{y}_i$. This factor through the localization

$$k[y_1, \ldots, y_d]_{(y_1,\ldots,y_d)} \to S$$

If the map was surjective, we would again be done. Unfortunately, it isn't usually surjective, but what is true and not hard to see is that the map surjects onto S/m^N for any N. This implies that it gives surjection of completions[2]

$$k[[y_1, \ldots, y_d]] \to \hat{S} := \varprojlim_N S/m^N$$

So now we have

$$\dim S = \dim \hat{S} \leq d$$

7.3 Regularity

Recall that noetherian local ring S is called *regular* if we have equality

$$\dim TS = \dim S$$

[2]See Atiyah-Macdonald for the basics about completions. One can view the completion as a way of making things "even more local" than ordinary localization. The intuition is that going to power series is like working with analytic neigbourhoods, which are finer than Zariski neighbourhoods.

A point $p \in X$ variety (scheme) is called nonsingular (regular) if the local ring $\mathcal{O}_{X,p}$ is regular. X is called nonsingular/regular if all its points have this property. Note that for varieties, the words are interchangeable. For schemes, nonsingularity is somewhat ambiguous because it might refer to regularity or a stronger condition called smoothness.

Theorem 7.3.1. *Let k be a field. If $X \subset \mathbb{A}^n_k$ is the closed subscheme defined by polynomials $f_1, \ldots, f_m$, $a \in X(k)$ is nonsingular if and only if*

$$\operatorname{rank}\left(\frac{\partial f_i}{\partial x_j}(a)\right) = n - \dim X$$

Proof. Let J be the Jacobian matrix above. Then $T_a X = \ker J(a)$ by (7.1). Therefore, the theorem follows from the rank-nullity theorem of linear algebra. □

Corollary 7.3.2. *The set of nonsingular points is open.*

Proof. This theorem and theorem 7.2.5 imply that a is nonsingular if and if

$$\operatorname{rank}(J(a)) \geq n - \dim X$$

The complementary condition is closed because it given by vanishing of minors of J. □

Note that this set might be empty. For example, this is the case for the double line $X = \operatorname{Spec} k[x, y]/(x^2)$. However, for varieties, it's a different story.

Theorem 7.3.3. *Let k be an algebraically closed field. Let $X \subset \mathbb{A}^n_k$ be the closed subvariety, or equivalently $\operatorname{Spec} k[x_1, \ldots x_n]/I$, where I is a prime ideal. Then the set of nonsingular points is nonempty and open.*

Corollary 7.3.4. *The same conclusion holds for quasiprojective varieties.*

We already know that this set is open, so we just have to prove that it is nonempty. We start with a special case.

Lemma 7.3.5. *The theorem holds for the hypersurface $X = V(f)$, where f is irreducible and nonconstant.*

Proof. We prove this by contradiction. So assume all points are singular. Then $f_{x_i}(a) = \partial f/\partial x_i(a) = 0$ for all $a \in X$ and i. Therefore $f_{x_i} \in (f)$ by the Nullstellensatz. Since $\deg f_{x_i} < \deg f$, we must have $f_{x_i} = 0$ as a polynomial, and this holds for each i. There two ways this can happen: Either the characteristic is 0, in which case f must be constant, or the characteristic is $p > 0$, and f is a pth power. In either case, this contradicts what we assumed about f. □

To finish the proof need the following.

Lemma 7.3.6. *If X is an affine variety, then it has a nonempty open set which is isomorphic to a nonempty open subset of some irreducible hypersurface.*

Proof. Let K be the function field of X. Recall that this is the field of fractions of $\mathcal{O}(X)$. Then K is finitely generated extension of k. By field theory, we can find a separating transcendence basis $x_1, \ldots x_n \in K$. This means that in the sequence

$$k \subset k(x_1, \ldots, x_n) \subset K$$

the first extension is purely transcendental, and the second is finite separable. The primitive element theorem allows us to express

$$K = k(x_1, \ldots, x_n, y)$$

for some element y. This is algebraic, so satisfies a nontrivial equation

$$f(x_1, \ldots, x_n, y) = \sum a_i(x_1, \ldots, x_n) y^i = 0$$

After clearing demoninators, we can assume that f is polynomial. Then $Y = V(f) \subset \mathbb{A}^{n+1}$ defines a hypersurface with the same function field as X, which is to say that X and Y are *birational*. The final step is observe that two birational varieties have open sets which are isomorphic. This is well known and standard. See for example p 26 of Hartshorne.

□

7.4 Exercises

Exercise 7.4.1.

1. *One can show that the category of schemes has products. Prove that $X \times Y(R) = X(R) \times Y(R)$. (This pretty formal. You won't need the construction of $X \times Y$.)*

A scheme G is called a group scheme (over $\mathbb{Z}$) if $G(R)$ is a group for each R, and each ring homomorphism $R \to S$ induces a group homomorphism $G(R) \to G(S)$. A stronger form of Yoneda's lemma than we stated shows that this equivalent to having morphisms $m : G \times G \to G$, $i : G \to G$ and $e : \operatorname{Spec} \mathbb{Z} \to G$ which makes G into group object in the category of schemes. This is similar to the way we defined algebraic groups.

Exercise 7.4.2.

2. *Let $G_a = \operatorname{Spec} \mathbb{Z}[T]$ (also called $\mathbb{A}^1_{\mathbb{Z}}$), $G_m = \operatorname{Spec} \mathbb{Z}[T, T^{-1}]$, and $\mu_n = \operatorname{Spec} \mathbb{Z}[T]/(T^n - 1)$. Show that these are group schemes.*

3. *Show that $\mathbb{A}^n_{\mathbb{Z}}$ is regular. Conclude the same for G_a and G_m.*

4. *Show that μ_n is not regular. (Look at primes dividing n.)*

5. *Prove that an algebraic group G is regular. (Hint: since G is a variety, it contains a maximal nonempty open set U. Use the group structure to show that $G = U$.)*