Curves, Surfaces, and Abelian Varieties

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April 27, 2017
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Chapter 1

Basic curve theory

1.1 Hyperelliptic curves

As all of us learn in calculus, integrals involving square roots of quadratic polynomials can be evaluated by elementary methods. For higher degree polynomials, this is no longer true, and this was a subject of intense study in the 19th century. An integral of the form

\[ \int \frac{p(x)}{\sqrt{f(x)}} \, dx \]  

(1.1)

is called elliptic if \( f(x) \) is a polynomial of degree 3 or 4, and hyperelliptic if \( f \) has higher degree, say \( d \).

It was Riemann who introduced the geometric point of view, that we should really be looking at the algebraic curve \( X^o \) defined by

\[ y^2 = f(x) \]

in \( \mathbb{C}^2 \). When \( f(x) = \prod_{i=0}^{d}(x - a_i) \) has distinct roots (which we assume from now on), \( X^o \) is nonsingular, so we can regard it as a Riemann surface or one dimensional complex manifold. \emph{Since surfaces will later come to mean two dimensional complex manifolds, we will generally refer to this as a (complex nonsingular) curve.} It is convenient to add points at infinity to make it a compact complex curve \( X \) called a (hyper)elliptic curve. One way to do this is to form the projective closure

\[ \bar{X}^o = \{ [x, y, z] \in \mathbb{P}^2 \mid F(x, y, z) = 0 \} \]

where

\[ F(x, y, z) = y^2 z^{d-2} - \prod(x - a_i z) = 0 \]

is the homogenization of \( y^2 - f(x) \). Unfortunately, \( \bar{X}^o \) will usually be singular. To see this, let’s switch to the affine chart \( y = 1 \). Then \( \bar{X}^o \) is given by

\[ z^{d-2} - \prod(x - a_i z) = 0 \]

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The partials vanish at \( x = 0, y = 0 \), as soon as \( d > 3 \).

We have to perform another operation on \( X^o \) which is called resolving singularities to obtain a nonsingular projective curve \( X \) containing \( X^o \). We give two constructions, both with their advantages and disadvantages. The first is a general procedure called normalization. If \( A \) is an integral domain with fraction field \( K \), its normalization or integral closure is the ring

\[
\tilde{A} = \{a \in K \mid \exists \text{ monic } f(t) \in A[t], f(a) = 0\} \supseteq A
\]

Suppose that \( X \) is an algebraic variety (or integral scheme) then \( X \) is obtained by gluing affine varieties \( U_i \) (or schemes) with coordinate rings \( A_i \). Let \( \tilde{U}_i = Spec \tilde{A}_i \). Then these can be glued to get a new variety/scheme \( \tilde{X} \) called the normalization. See Mumford’s Red Book for details. The normalization comes with a morphism \( \tilde{X} \to X \).

Here is an example. Let \( A = k[x, y]/(y^2 - x^3) \) be the coordinate ring of a cusp over a field \( k \). Then \( y/x \in \tilde{A} \) because it satisfies \( t^2 - x = 0 \). With more work, we can see that \( \tilde{A} = k[y/x] \). In general, by standard commutative algebra

**Theorem 1.1.1.** If \( A \) is a one dimensional integral domain, then \( \tilde{A} \) is Dedekind domain; in particular the localizations of \( \tilde{A} \) at maximal ideals are discrete valuation rings and therefore regular.

**Corollary 1.1.2.** If \( X \) is a curve i.e. one dimensional variety, then \( \tilde{X} \) is a nonsingular curve with the same function field as \( X \).

Returning to the original problem. Given \( \bar{X}^o \) as above, \( X \) can be simply be taken to be it’s normalization. Unfortunately, this process is not very geometric. So we briefly describe another procedure. The blow up of the affine plane \( \mathbb{A}^2_k \) at the origin is the quasiprojective variety

\[
B = Bl_0 \mathbb{A}^2 = \{(v, \ell) \in \mathbb{A}^2 \times \mathbb{P}^1 \mid v \in \ell\}
\]

\[
= \{(x, y, [X, Y]) \mid \mathbb{A}^2 \times \mathbb{P}^1 \mid xY = XY\}
\]

This comes with a projection \( \pi : B \to \mathbb{A}^2 \) which is an isomorphism over \( \mathbb{A}^2 - \{0\} \). The blow of the projective plane can be defined as the closure of \( B \) in \( \mathbb{P}^2 \times \mathbb{P}^1 \). Using the Segre embedding, we can see that this is a projective variety.

Given a curve \( C \subset \mathbb{A}^2 \) (or \( \mathbb{P}^2 \)), the closure \( C_1 \) of \( \pi^{-1}C - \{0\} \) is the blow up of \( C \). This is also called the strict transform of \( C \). Let’s calculate this for the cusp \( y^2 - x^3 = 0 \) in \( \mathbb{A}^2 \). Then \( B \) is covered by open sets where \( X = 1 \) and \( Y = 1 \). The intersection of \( C_1 \) with \( X = 1 \) is the irreducible component of

\[
\{y^2 = x^3, xY = y\} \Leftrightarrow \{x(Y^2 - x) = 0, y = xY\}
\]

dominating \( C \). This is the locus \( Y^2 - x = 0 \), which is nonsingular. In general, the process many steps. For the example, \( y^2 = x^5 \), the first step produces \( C_1 : Y^2 - x^3 = 0 \), which is “less singular” than before. Blowing it up a second time at \( x = Y = 0 \), yields a nonsingular curve \( C_2 \).
Theorem 1.1.3. Given a curve \( X \), after finite number of blow ups, we obtain a nonsingular curve \( \tilde{X} \rightarrow X \). This coincides with the normalization.

Corollary 1.1.4. \( \tilde{X} \) can be embedded into a projective space (generally bigger than \( \mathbb{P}^2 \)).

1.2 Topological genus

Let’s work over \( \mathbb{C} \). Then a nonsingular projective curve \( X \subset \mathbb{P}^n_\mathbb{C} \) can be viewed as complex submanifold and therefore a Riemann surface.\(^1\) In particular, \( X \) can be viewed as a compact \( C^\infty \) (real) dimensional manifold. The fact that \( X \) is complex implies that it is orientable. Before describing the classification, recall that the given two connected \( n \)-manifolds \( X_1, X_2 \), their connected sum \( X_1 \# X_2 \) is obtained by removing an \( n \)-ball from both and joining them by the cylinder \( S^{n-1} \times [0,1] \).

Theorem 1.2.1. Any compact connected orientable 2-manifold is homeomorphic to either the two sphere \( S^2 \) or connected sum of \( g \) 2-tori for some integer \( g > 0 \).

The integer \( g \) is called the genus. However, we will refer to as the topological genus temporarily until we have established all the properties. We set \( g = 0 \) in the case of \( S^2 \). Let’s relate this to more familiar invariants. Recall that the Euler characteristic \( e(X) \) of a (nice) topological space is the alternative sum of Betti numbers. When the space admits a finite triangulation, then it is number of vertices minus the number of edges plus ... We can triangulate \( S^2 \) as a tetrahedron, therefore \( e(S^2) = 4 - 6 + 4 = 2 \). In general, we can compute the Euler characteristic using the following inclusion-exclusion formula:

Proposition 1.2.2. If \( X = U \cup V \) is a union of open sets, \( e(X) = e(U) + e(V) - e(U \cap V) \).

Proof. In general, given an exact sequence of vector spaces

\[ \ldots V^i \rightarrow V^{i+1} \rightarrow \ldots \]

\[ \sum (-1)^i \dim V^i = 0 \]

Now apply this to the Mayer-Vietoris sequence

\[ \ldots H^i(X) \rightarrow H^i(U) \oplus H^i(V) \rightarrow H^i(U \cap V) \rightarrow \ldots \]

where \( H^i(X) = H^i(X, \mathbb{C}) \).

Proposition 1.2.3. Given a compact orientable surface \( X \) of genus \( g \)

(a) \( e(X) = 2 - 2g \)

\(^1\)Recall that a Riemann surface is the same thing as a one dimensional complex manifold.
(b) If \( D \subset X \) is a disk, \( e(X - D) = 1 - 2g \)

Proof. By the previous proposition have \( e(X) = e(D') + e(X - D) - e(D' \cap (X - D)) \) where \( D' \supset D \) is a slightly larger disk. Since, Betti numbers, and therefore Euler characteristic, is invariant under homotopy, \( e(D') = e(pt) = 0 \) and \( e(D' \cap (X - D)) = e(S^1) = 0 \). Therefore \( e(X - D) = e(X) - 1 = 1 - 2g \) assuming (a).

We prove (a) by induction. The \( g = 0 \) case is clear. We can write \( X = Y \# T \), where \( Y \) has genus \( g - 1 \). Therefore

\[
e(X) = e(Y - D) + e(T - D) - e(S^1 \times [0, 1]) = 1 - 2(g - 1) - 1 = 2 - 2g
\]

We can use this to compute the genus for our (hyper)elliptic curve \( X \) obtained from \( y^2 = f(x) \) as before. We have holomorphic map \( p : X \to \mathbb{P}^1 \) extending the projection of the affine curve to the \( x \)-axis. Note that this map is 2 to 1 for all but finitely many points called the branch points. These consist of the zeros of \( f \) and possibly \( \infty \). Let \( r \) be the number of these points. Either \( r = \deg f \) or \( \deg f + 1 \). Let us triangulate \( \mathbb{P}^1 \), making sure to include the branch points among the vertices and no edge connects two branch points. Let \( V = r + V' \) be the number of vertices, \( E \) the number of edges and \( F \) the number of faces. Since \( \mathbb{P}^1 \) is \( S^2 \) as topological space,

\[
V - E + F = e(\mathbb{P}^1) = 2
\]

Take the preimage of this triangulation under \( p \). This gives a triangulation of \( X \), with \( r + 2V' \) vertices, \( 2E \) edges and \( 2F \) faces. Therefore \( g \) is the genus of \( X \), we have

\[
2 - 2g = r + 2V' - 2E + 2F = 2(V - E + F) - R = 4 - r
\]

Thus

\textbf{Proposition 1.2.4.}

\[
g = \frac{1}{2} r - 1
\]

Note that this forces \( R \) to be even, and this allows us to conclude that

\[
r = \begin{cases} 
\deg f & \text{if } \deg f \text{ is even} \\
\deg f + 1 & \text{otherwise}
\end{cases}
\]

This result generalizes. Let \( f : X \to Y \) be a surjective holomorphic map between compact curves. A point \( p \in X \) is called a ramification point if the derivative of \( p \) vanishes at it. An image of a ramification point will be called a branch point. (Some people reverse the terminology.) There are a finite number of branch points \( B \). After removing these we get a finite sheeted covering space \( X - f^{-1}B \to Y - B \). The number of sheets is called the degree of \( p \). Let us
denote this by $d$. So for each nonbranch point $|f^{-1}(q)| = d$. Now suppose that $q$ is a branch point. Choose a small disk $D$ centered at $q$. The premiage $f^{-1}(D)$ is a union of disks $D_i$ centered at $p_i \in f^{-1}(q)$. We can choose coordinates so $D_i \to D$ is given by $y = x^{e_i}$. The exponents $e_i = e(p_i)$ are called ramification indices. After perturbing $q$ slightly, $p_i$ splits into a union of $e_i$ points. Therefore

$$\sum e_i = d$$

By the same sort of argument as before, we get

**Theorem 1.2.5** (Riemann-Hurwitz). If $g(X)$ and $g(Y)$ denote the genera of $X$ and $Y$,

$$2(g(X) - 2) = 2d(g(X) - 1) + \sum_{p \in X} e(p) - 1$$

### 1.3 Degree of the canonical divisor

While the previous description of the genus is pretty natural. It is topological rather than algebro-geometric. We give an alternative description which actually works over any field.

Let $X$ be a compact curve. A divisor on $X$ is finite formal sum $D = \sum n_i p_i$, $n_i \in \mathbb{Z}$, $p_i \in X$. The degree $degD = \sum n_i$. If $f$ is a nonzero meromorphic function, the associated principal divisor

$$div(f) = \sum ord_p(f) p$$

where $ord_p(f)$ is the order of the zero $f$ at $p$ or minus the order of the pole.

**Proposition 1.3.1.** $deg \, div(f) = 0$

**Proof.** Let $p_j$ be the set of zeros and poles of $f$, and let $D_j$ be a small disk centered at $p_j$. We can express $ord_p(f)$ as the residue of $df/f$ at $p_j$. Thus by Stokes

$$deg \, div(f) = \frac{1}{2\pi i} \sum \int_{\partial D_j} \frac{df}{f} = -\frac{1}{2\pi i} \int_{X-\cup D_j} d \left( \frac{df}{f} \right) = 0$$

Given a nonzero meromorphic 1-form $\alpha$, set

$$div(\alpha) = \sum ord_p(\alpha) p$$

Such a divisor is called a canonical divisor. Inspite of the similarity of notation, it is usually not principal. However any two canonical divisors differ by a principal divisor. Consequently the degree of a canonical divisor depends only on $X$. We write it as $deg K_X$. For this to make sense, we need to show that a nonzero constant meromorphic form actually exists. Let us avoid the issue for now by restricting our attention to smooth projective curves.\(^2\)

\(^2\)In fact, as we will see later, all compact Riemann surfaces are of this form.
Lemma 1.3.2. The degree of a canonical divisor on \( \mathbb{P}^1 \) is \(-2\).

Proof. Let \( z \) be the standard coordinate on \( \mathbb{C} \). We can use \( \zeta = z^{-1} \) as the coordinate around \( \infty \). Since \( dz = d\zeta^{-1} = -\zeta^{-2}d\zeta \). This proves the lemma. \( \square \)

Theorem 1.3.3 (Riemann-Hurwitz II). Let \( f : X \to Y \) be a branched cover of degree \( d \), then

\[
\deg K_X = \deg K_Y + \sum_{p \in X} e(p) - 1
\]

Proof. Let \( \alpha \) be a nonzero meromorphic form on \( Y \). We can assume, after multiplying \( \alpha \) by a suitable meromorphic function, that set of zeros or poles are disjoint from the branch points. If \( q \) is a zero or pole of \( \alpha \) or order \( n \), then \( f^*\alpha \) has zero or pole at each \( p \in f^{-1}q \) of order \( n \). A a branch point \( q \), locally \( \alpha = u(y)dy \), where \( u \) is holomorphic and nonzero at \( q \). Then \( f^*\alpha = e_c u(x^c) x^{c-1} dy \) Therefore \( \deg(div(f^*\alpha)) \) is given by the right side of the formula, we are trying to prove. \( \square \)

Corollary 1.3.4. For any nonsingular projective curve, \( \deg K_X = 2g - 2 \).

Observe that this says that \( \deg K_X = -e(X) \). A more conceptual explanation can be given using Chern classes discussed later. The Chern number of the tangent bundle is \( e(X) \), whereas \( \deg K_X \) is the Chern number of the cotangent bundle.

1.4 Line bundles

Given \( X \) be as above. The divisors from a group \( \text{Div}(X) \), and the principal divisors form a subgroup \( \text{Princ}(X) \). The divisor class group

\[
\text{Cl}(X) = \text{Div}(X)/\text{Princ}(X)
\]

We have a surjective homorphism

\[
\deg : \text{Cl}(X) \to \mathbb{Z}
\]

induced by the degree. To understand the structure of the kernel \( \text{Cl}^0(X) \), we bring in modern tools. Recall that a holomorphic line bundle is a complex manifold \( L \) with a holomorphic map \( \pi : L \to X \) which “locally looks like” the projection \( \mathbb{C} \times X \to X \). More precisely, there exists an open cover \( \{U_i\} \) and holomorphic isomorphisms \( \psi_i : \pi^{-1}U_i \to U_i \times \mathbb{C} \), which are linear on the fibres. Such a choice called a local trivialization. Given a line bundle \( \pi : L \to X \), the sheaf of sections

\[
\mathcal{L}(U) = \{ f : U \to \pi^{-1} \mid f \text{ holomorphic and } \pi \circ f = \text{id} \}
\]

is a locally free \( \mathcal{O}_X \)-module of rank 1. Conversely, any such sheaf arises this way from a line bundle which is unique up to isomorphism. We will therefore identify the two notions.
To be a bit more explicit, fix $L$ with a local trivialization. Set $U_{ij} = U_i \cap U_j$ etc., and let

$$\Phi_{ij} : U_{ij} \times \mathbb{C} \xrightarrow{\psi_i^{-1}} \pi^{-1}U_{ij} \xrightarrow{\psi_j} U_{ij} \times \mathbb{C}$$

It is easy to see that this collection of maps satisfies the 1-cocycle conditions:

$$\Phi_{ik} = \Phi_{ij}\Phi_{jk}, \text{ on } U_{ijk}$$

$$\Phi_{ij} = \Phi_{ji}^{-1}$$

$$\Phi_{ii} = id$$

We can decompose

$$\Phi_{ij}(x) = (x, \phi_{ij}(x))$$

where $\phi_{ij} : U_{ij} \to \mathbb{C}^*$ is a collection of holomorphic functions satisfying the 1-cocycle conditions. This determines a Čech cohomology class in $H^1(\{U_i\}, \mathcal{O}_X^*)$. Conversely, such a class determines a line bundle. We can summarize everything as follows.

**Theorem 1.4.1.** There is an bijection between

1. the set of isomorphism classes of line bundles on $X$
2. the set of isomorphism classes of rank one locally free $\mathcal{O}_X$-modules
3. $H^1(X, \mathcal{O}_X^*)$

The last set is of course a group, called the Picard group. It is denoted by $Pic(X)$. The group operation for line bundles, or sheaves is just tensor product. The inverse of the dual line bundle $L^{-1} = \text{Hom}(L, \mathcal{O}_X)$. We have an exact sequence of sheaves

$$0 \to \mathbb{Z} \to \mathcal{O}_X \xrightarrow{2\pi i} \mathcal{O}_X^* \to 1$$

This yields a long exact sequence

$$H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}_X) \to Pic(X) \xrightarrow{c_1} H^2(X, \mathbb{Z})$$

The last map is called the first Chern class. This is can viewed as an integer because $H^2(X, \mathbb{Z}) \cong \mathbb{Z}$. The kernel of $c_1$ is denoted by $Pic^0(X)$. We will show later that $H^1(X, \mathbb{Z})$ sits as a lattice in $H^1(X, \mathcal{O}_X)$. Therefore $Pic^0(X)$ has the structure of a complex torus.

We can relate this to the divisor class group. Let $K = \mathbb{C}(X)$ be the field meromorphic functions on $X$. Given $D = \sum n_i p_i \in \text{Div}(X)$, define the sheaf

$$\mathcal{O}_X(D)(U) = \{ f \in K \mid ord_p f + n_i \geq 0 \}$$

This is a locally principal fractional ideal sheaf, and therefore a line bundle. A straight forward verification shows that
Lemma 1.4.2.
\[ O_X(D_1) \otimes O_X(D_2) \cong O_X(D_1 + D_2) \]
and
\[ O_X(D) \cong O_X \]
when \( D \) is principal.

Therefore map \( D \mapsto O_X(D) \) induces a homomorphism \( Cl(X) \to Pic(X) \).

Theorem 1.4.3. The above map gives an isomorphism \( Cl(X) \cong Pic(X) \).

Proof of injectivity. Let \( D = \sum n_i p_i \). Suppose that \( O_X(D) \cong O_X \). Then the section 1 on the right corresponds to a global section \( f \in H^0(X, O_X(D)) \) which generates all the stalks of \( O(D)_p \). We can identify \( O(D)_p \) with \( x^{-n_i} O_{p_i} \), where \( x \) is a local parameter. For \( f \) to generate \( x^{-n_i} O_{p_i} \), we must have \( ord_{p_i} f = -n_i \). Therefore \( D = \text{div}(f^{-1}) \). Thus the homomorphism \( Cl(X) \to Pic(X) \) is injective. We will prove surjectivity in the next section.

We have an isomorphism \( H^2(X, \mathbb{Z}) \cong \mathbb{Z} \), therefore \( c_1(\mathcal{L}) \) can be viewed as a number. Let us explain how to compute it in terms of a 1-cocycle \( \phi_{ij} \) for \( \mathcal{L} \) on \( \mathcal{U} = \{ U_i \} \). We have another sequence
\[ 0 \to \mathbb{C} \to O_X \xrightarrow{d} \Omega^1_X \to 0 \]
We can map the exponential sequence to this
\[ 0 \to \mathbb{Z} \to O_X \xrightarrow{\frac{1}{2\pi i} d \log} O^*_X \to 1 \]
\[ 0 \to \mathbb{C} \to O_X \xrightarrow{\frac{1}{2\pi i} d \log} \Omega^1_X \to 0 \]
This shows that \( c_1 \) can be factored as
\[ H^1(X, O^*_X) \to H^1(X, \Omega^1_X) \]
composed with the connecting map
\[ H^1(X, \Omega^1_X) \to H^2(X, \mathbb{C}) \]
We need to make the last part explicit. Let \( \alpha_{ij} \in Z^1(\mathcal{U}, \Omega^1) \) be the cocycle \( \frac{1}{2\pi i} d \log \phi_{ij} \). We can view this as a 1-cocycle in \( Z^1(\mathcal{U}, \mathcal{E}^1_X) \), where \( \mathcal{E}^1_X \) are \( C^\infty \) 1-forms. But we know that \( H^1(X, \mathcal{E}^1_X) = 0 \) because \( \mathcal{E}^1_X \) is soft. Therefore we can find \( C^\infty \) 1-forms \( \alpha_i \) such that \( \alpha_{ij} = \alpha_i - \alpha_j \). Since \( d\alpha_{ij} = d\alpha_{ij} = 0 \), \( \beta = d\alpha_i \) is a globally defined 2-form. The de Rham class of \( \beta \) is precisely what we are after. Under the isomorphism
\[ H^2(X, \mathbb{C}) \cong \mathbb{C} \]
given by integration,
\[ c_1(\mathcal{L}) = \int_X \beta \]
This is really only a first step. A more satisfying answer is given by the next theorem.

**Theorem 1.4.4.** After identifying \( H^2(X, \mathbb{Z}) = \mathbb{Z}, c_1(\mathcal{O}_X(D)) = \deg D \).

**Proof.** It’s enough to prove that \( c_1(\mathcal{O}_X(-p)) = -1 \) for every \( p \in X \). Let \( U_0 \) be a coordinate disk around \( p \) with coordinate \( z \), and let \( U_1 = X - \{p\} \). \( \mathcal{O}_X(-p) \subset \mathcal{O}_X \) is the ideal of \( p \). On \( U_0 \), \( z \) gives a trivializing section; on \( U_1 \) we can use 1. The change of basis function \( \phi_{01} = z^{-1} \) is the cocycle for \( \mathcal{O}(p) \). So our task is to compute \( \int_X \beta \), where \( \beta \) is obtained from \( \phi_{01} = z^{-1} \) by the above process. We split \( X = E \cup B \), where \( B = U_0 \) is the closed disk and \( E = X - U_0 \). Let \( C = \partial B \) oriented counterclockwise around \( p \). By Stokes and the residue theorem
\[
\int_X \beta = \int_E \beta + \int_B \beta = \int_E d\alpha_1 + \int_B d\alpha_0 = \int_C \alpha_0 - \alpha_1 = \frac{-1}{2\pi i} \int_C \frac{dz}{z} = -1
\]
This concludes the proof. \( \square \)

**Corollary 1.4.5.** \( Cl^0(X) \cong Pic^0(X) \).

### 1.5 Serre duality

Let \( X \) be a connected compact complex curve of genus \( g \). Let \( \mathcal{O}_X \) denote the sheaf of holomorphic functions and let \( \Omega^1_X \) denote the sheaf of holomorphic 1-forms on \( X \). We have an exact sequence of sheaves
\[
0 \to \mathbb{C}_X \to \mathcal{O}_X \xrightarrow{d} \Omega^1_X \to 0
\]
which gives a long exact sequence
\[
0 \to H^0(X, \mathbb{C}) \to H^0(X, \mathcal{O}_X) \to H^0(X, \Omega^1_X) \to H^1(X, \mathbb{C})
\]
Since \( X \) is connected \( H^0(X, \mathbb{C}) = H^0(X, \mathcal{O}_X) = \mathbb{C} \), where the last equality follows from the maximum principle. Therefore we have an injection
\[
0 \to H^0(X, \Omega^1_X) \to H^1(X, \mathbb{C})
\]
In particular, \( \dim H^0(X, \Omega^1_X) \leq 2g \). In fact, we do much better. If we identify \( H^1(X, \mathbb{C}) \) with de Rham cohomology, then we have a pairing
\[
H^1(X, \mathbb{C}) \times H^1(X, \mathbb{C}) \to \mathbb{C}
\]
given by
\[ \langle \alpha, \beta \rangle = \int_X \alpha \wedge \beta \] (1.2)

A suitably strong form of Poincaré duality shows that this is a nondegenerate pairing. It is clearly also skew symmetric i.e.

\[ \langle \alpha, \beta \rangle = -\langle \beta, \alpha \rangle \]

We quote the following fact from linear algebra

**Theorem 1.5.1.** If \( V \) is finite dimensional vector with a nondegenerate skew symmetric pairing \( \langle , \rangle \), \( \dim V \) is even. If \( W \subset V \) is subspace such that \( \langle \alpha, \beta \rangle = 0 \) for all \( \alpha, \beta \in W \) (\( W \) is called isotropic), then \( \dim W \leq \frac{1}{2} \dim V \).

**Corollary 1.5.2.** \( \dim H^0(X, \Omega^1_X) \leq g \).

**Proof.** The formula (1.2) makes it clear that this is isotropic. \( \Box \)

Recall last semester, we constructed a Dolbeault resolution

\[
0 \to \mathcal{O}_X \to \mathcal{E}^{00}_X \overset{\bar{\partial}}{\to} \mathcal{E}^{01}_X \to 1
\]

where \( \mathcal{E}^{00}_X \) (resp \( \mathcal{E}^{(0,1)}_X \)) is the sheaf of \( C^\infty \) functions (resp locally \( C^\infty \) multiples of \( d\bar{z} \)). \( \bar{\partial}f \) is \((0,1)\)-part of \( df \). Since this gives a soft resolution of \( \mathcal{O}_X \), we can use this to compute sheaf cohomology

\[
H^0(X, \mathcal{O}_X) = \ker[\mathcal{E}^{00}_X(X) \overset{\bar{\partial}}{\to} \mathcal{E}^{01}_X(X)]
\]

\[
H^1(X, \mathcal{O}_X) = \text{coker}[\mathcal{E}^{00}_X(X) \overset{\bar{\partial}}{\to} \mathcal{E}^{01}_X(X)]
\]

\[
H^i(X, \mathcal{O}_X) = 0, i \geq 2
\]

Suppose that \( \alpha \in \Omega^1(X) \) and \( \beta \in \mathcal{E}^{(0,1)}(X) \). Define

\[ \langle \alpha, \beta \rangle = \int_X \alpha \wedge \beta \]

as before.

**Lemma 1.5.3.** This gives a well defined pairing

\[ \langle , \rangle : H^0(X, \Omega^1_X) \times H^1(X, \mathcal{O}_X) \to \mathbb{C} \]

**Proof.** It is enough to show that \( \langle \alpha, \bar{\partial}f \rangle = 0 \). This follows from Stokes' theorem because \( \alpha \wedge \bar{\partial}f = \pm d(f\alpha) \). \( \Box \)

**Theorem 1.5.4** (Serre duality I). The above pairing is perfect, i.e. it induces an isomorphism

\[ H^1(X, \mathcal{O}_X) \cong H^0(X, \Omega^1_X)^* \]
Corollary 1.5.5. \( \dim H^0(X, \Omega_X^1) = \dim H^1(X, \mathcal{O}_X) = g \)

Proof. The theorem gives \( \dim H^0(X, \Omega_X^1) = \dim H^1(X, \mathcal{O}_X) \). Call his number \( h \). Earlier we saw that \( h \leq g \). The exact sequence

\[
0 \to H^0(X, \Omega_X) \to H^1(X, \mathbb{C}) \to H^1(X, \mathcal{O}_X)
\]

forces \( 2g \leq 2h \).

What we stated above is really a special case of Serre duality:

Theorem 1.5.6 (Serre duality II). If \( \mathcal{L} \) is a line bundle then there is a natural pairing inducing an isomorphism

\[
H^1(X, \mathcal{L}) \cong H^0(X, \Omega_X^1 \otimes \mathcal{L}^{-1})^*
\]

We can use this finish the proof of theorem 1.4.3. It remains to prove that the map \( Cl(X) \to Pic(X) \) is surjective that is:

Proposition 1.5.7. Any line bundle is isomorphic to \( \mathcal{O}_X(D) \) for some divisor \( D \).

We first need a few lemmas which are useful by themselves.

Lemma 1.5.8. If \( \deg \mathcal{L} < 0 \), then \( H^0(X, \mathcal{L}) = 0 \).

Proof. A nonzero section would correspond to a nonzero map \( \sigma : \mathcal{O}_X \to \mathcal{L} \). Dualizing gives a nonzero map

\[
\sigma^* : \mathcal{L}^{-1} \to \mathcal{O}_X
\]

Consider \( \mathcal{K} = \ker \sigma^* \). Then \( \mathcal{K} \otimes \mathcal{L} \subset \mathcal{O}_X \) is ideal sheaf. It must by either 0 or of the form \( \mathcal{O}_X(-D) \) for some effective divisor \( D \) (effective means that the coefficients are nonnegative). It follows that either \( \mathcal{K} = \mathcal{L}^{-1}(-D) := \mathcal{L}^{-1} \otimes \mathcal{O}(-D) \) if it isn’t zero. If \( D = 0 \), then \( \sigma^* = 0 \) which is impossible. If \( D > 0 \), then \( \mathcal{O}_X \) would contain the nonzero torsion sheaf \( \mathcal{L}^{-1}/\mathcal{K} \) which is also impossible. Therefore \( \mathcal{K} = 0 \), and consequently \( \sigma^* \) is injective. Thus \( \mathcal{L}^{-1} \subset \mathcal{O}_X \) can be identified with \( \mathcal{O}_X(-D) \) with \( D \) effective. This implies \( \deg \mathcal{L} = \deg D \geq 0 \), which is a contradiction.

Corollary 1.5.9. If \( \deg \mathcal{L} > 2g - 2 \) then \( H^1(X, \mathcal{L}) = 0 \).

Proof. By corollary 1.3.4 \( \deg \Omega_X^1 \otimes \mathcal{L}^{-1} = 2g - 2 - \deg \mathcal{L} < 0 \). This implies that \( H^0(X, \Omega_X^1 \otimes \mathcal{L}^{-1}) = 0 \).

Lemma 1.5.10. If \( \deg \mathcal{L} > 2g - 1 \), then \( H^0(X, \mathcal{L}) \neq 0 \)

Proof. Choose \( p \in X \), and let \( \mathbb{C}_p \) be the skyscraper sheaf

\[
\mathbb{C}_p(U) = \begin{cases} 
\mathbb{C} & \text{if } p \in U \\
0 & \text{otherwise}
\end{cases}
\]
Then we have an exact sequence

\[ 0 \to \mathcal{O}_X(-p) \to \mathcal{O}_X \to \mathbb{C}_p \to 0 \]

Tensoring with \( \mathcal{L} \) gives a sequence

\[ 0 \to \mathcal{L}(-p) \to \mathcal{L} \to \mathbb{C}_p \to 0 \]

because \( \mathbb{C}_p \otimes \mathcal{L} \cong \mathbb{C}_p \). Therefore we get an exact sequence

\[ H^0(X, \mathcal{L}) \to \mathbb{C} \to H^1(X, \mathcal{L}(-p)) = 0 \]

This proves the lemma.

**Proof of proposition 1.5.7.** Let \( \mathcal{L} \) be a line bundle. By the previous \( \mathcal{L}(F) \) has a nonzero section for \( \deg F \gg 0 \). This gives a nonzero map \( \mathcal{O}_X \to \mathcal{L}(D) \) or dually a nonzero map \( \mathcal{L}^{-1}(-F) \to \mathcal{O}_X \). Arguing as above, we see that \( \mathcal{L}^{-1}(-F) = \mathcal{O}_X(-E) \) for some effective divisor \( E \). Therefore \( \mathcal{L} \cong \mathcal{O}_X(E - F) \).

### 1.6 Harmonic forms

We outline the proof of theorem 1.5.4. We start with a seemingly unrelated problem. Recall that de Rham cohomology

\[ H^1 \text{dR}(X, \mathbb{C}) = \{ \alpha \in \mathcal{E}^1(X) | d\alpha = 0 \} \]

So an element of it is really an equivalence class. Does such a class have a distinguished representative? We can answer the analogous problem in finite dimensional linear algebra by the method of least squares: given a finite dimensional inner product space \( V \) with a subspace \( W \), there is a subspace \( V/W \) with a smallest norm. We introduce a norm on our space as follows. In local analytic coordinates \( z = x + yi \), define \( *dx = dy, *dy = -dx \). This is amounts to multiplication by \( i \) in the cotangent plane, so it is globally well defined operation. We extend this to \( \mathbb{C} \)-linear operator. Then

\[ (\alpha, \beta) = \int_X \alpha \wedge \bar{\beta} \]

gives an inner product on \( \mathcal{E}^1(X) \) and therefore a norm \( ||\alpha||^2 = (\alpha, \alpha) \).

**Theorem 1.6.1** (Hodge theorem). Every cohomology class has unique representative which minimizes norm.

We want to explain the uniqueness part. First we need to understand the norm minimizing condition in more explicit terms. We define a 1-form \( \alpha \) to be harmonic if \( d\alpha = d(*\alpha) = 0 \). (This is a solution of a Laplace equation, as the terminology suggests, but we won’t need this.)
Lemma 1.6.2. A harmonic form is the unique element of smallest norm in its cohomology class. Conversely, if a closed form minimizes norm in its class, then it is harmonic.

Proof. For simplicity, let’s prove this for real forms.

As a first step, we can establish the identity
\[ \langle df, \alpha \rangle = -\int_X f d^* \alpha \] (1.3)
by observing that
\[ \int_X d(f \wedge \ast \alpha) = 0 \]
by Stokes’ and then expanding this out. Therefore
\[ \|\alpha + df\|^2 = \|\alpha\|^2 + 2\langle df, \alpha \rangle + \|df\|^2 = \|\alpha\|^2 - 2\int_X f d^* \alpha + \|df\|^2 \] (1.4)

If \( \alpha \) is harmonic, then
\[ \|\alpha + df\|^2 = \|\alpha\|^2 + \|df\|^2 > \|\alpha\|^2 \]
when \( df \neq 0 \).

Now suppose that \( \|\alpha\|^2 \) is minimal in its class. Then for any \( f \), we must have
\[ \frac{d}{dt}\|\alpha + t\|^2|_{t=0} = 0 \]
Using (1.4) we conclude that
\[ \int_X f d^* \alpha = 0 \]
Since \( f \) is arbitrary, we must have \( d^* \alpha = 0 \).

\[ \square \]

A special case of the uniqueness statement, which follows directly from (1.3), is

Corollary 1.6.3. \( 0 \) is the only exact harmonic form.

The proof of the existence statement for theorem 1.6.1 is where most of the analytic subtleties lie. We won’t give the proof, but refer the reader to Griffiths–Harris or Wells. The only thing we want to observe is that the proof yields an extension of theorem 1.6.1 which applies to non closed forms.

Theorem 1.6.4 (Hodge theorem II). Any form \( E^1(X) \) can be uniquely decomposed into a sum \( \beta + df + \ast dg \), where \( \beta \) is harmonic and \( f, g \) are \( C^\infty \) functions.

Proposition 1.6.5.
(a) A harmonic 1-form is a sum of a (1, 0) harmonic form and (0, 1) harmonic form.

(b) A (1, 0)-form is holomorphic if and only if it is closed if and only it is harmonic.

(c) A (0, 1)-form is harmonic if and only if it is antiholomorphic i.e. its complex conjugate is holomorphic.

Proof. If \( \alpha \) is a harmonic 1-form, then 
\[
\alpha = \alpha' + \alpha'',
\]
where \( \alpha' = \frac{1}{2} (\alpha + i * \alpha) \) is a harmonic (1, 0)-form and 
\( \alpha'' = \frac{1}{2} (\alpha - i * \alpha) \) is a harmonic (0, 1)-form.

If \( \alpha \) is (1, 0), then \( d\alpha = \bar{\partial} \alpha \). This implies the first half (b). For the second half, use the identity
\[
\ast dz = \ast(dx + idy) = dy - idx = -idz
\]
Finally, note that the harmoncity condition is invariant under conjugation, so the (c) follows from (b).

Proposition 1.6.6. \( H^1(X, O_X) \) is isomorphic to the space of antiholomorphic forms.

Proof. Let \( H \subset \mathcal{E}^{01}(X) \) denote the space of antiholomorphic forms. We will show that 
\[
\pi : H \to \mathcal{E}^{01}(X)/\text{im} \bar{\partial}
\]
is an isomorphism. Suppose that \( \alpha \in \mathcal{E}^{01}(X) \). By theorem 1.6.4, we may choose a harmonic form \( \beta \) such that 
\[
\beta = \alpha + df + *dg
\]
for some \( f, g \in C_\infty(X) \). Then the (0, 1) part of \( \beta \) gives an element \( \beta' \in H \) such that 
\[
\beta' = \alpha + \bar{\partial}(f + i g).
\]
This shows that \( \pi \) is surjective.

Suppose that \( \alpha \in \ker \pi \). Then \( \alpha = \bar{\partial}f \) for some \( f \). Therefore \( \alpha + \bar{\alpha} = df \). Consequently \( \alpha + \bar{\alpha} \) is exact and harmonic, so \( \alpha + \bar{\alpha} = 0 \). This implies \( \alpha = 0 \).

Corollary 1.6.7. \( \alpha \mapsto \bar{\alpha} \) gives a conjugate linear isomorphism
\[
H^0(X, \Omega^1_X) \cong H^1(X, O_X)
\]
Proof of theorem 1.5.4. We have a linear map
\[
\sigma : H^0(X, \Omega^1_X) \to H^1(X, O_X)^*
\]
which assigns to \( \alpha \) the functional \( \langle \alpha, - \rangle \). If \( \alpha \in H^0(X, \Omega^1_X) \) is nonzero then
\[
\sigma(\alpha)(\bar{\alpha}) = \int_X \alpha \wedge \bar{\alpha} \neq 0
\]
Therefore \( \sigma \) is injective. Since these spaces have the same dimension, \( \sigma \) is an isomorphism.

The general Serre duality can also be proved by a similar method.
1.7 Riemann-Roch

Let \( h^i(L) = \dim H^i(X, L) \), and \( \chi(L) = h^0(L) - h^1(L) \).

**Theorem 1.7.1** (Riemann-Roch). If \( X \) is compact curve of genus \( g \), and \( D \) is a divisor

\[
\chi(O_X(D)) = \deg D + 1 - g
\]

**Proof.** Let \( D = \sum n_i p_i \). We prove this by induction on the “mass” \( M(D) = \sum |n_i| \). When \( M(D) = 0 \), then this is just the equality

\[
\chi(O_X) = 1 - g
\]

We follow from the facts \( h^0(O_X) = 1 \), \( h^1(O_X) = g \) established earlier.

Given \( p \), we have an exact sequence

\[
0 \to O_X(-p) \to O_X \to \mathbb{C}_p \to 0
\]

Tensoring by \( O_X(D) \) gives

\[
0 \to O_X(D - p) \to O_X(D) \to \mathbb{C}_p \to 0
\]

Observe that \( H^0(X, \mathbb{C}_p) = \mathbb{C} \) by definition and \( H^1(X, \mathbb{C}_p) = 0 \) because \( \mathbb{C}_p \) is flasque. Thus

\[
\chi(O_X(D)) = \chi(O_X(D - p)) + \chi(\mathbb{C}_p) = \chi(O_X(D - p)) + 1
\]

Therefore

\[
\chi(O(D)) - \deg D = \chi(O(D - p)) - \deg(D - p)
\]

or by changing variable and writing the formula backwards

\[
\chi(O(D)) - \deg D = \chi(O(D + p)) - \deg(D + p)
\]

Since we can choose \( p \) so that \( M(D \pm p) < M(D) \). One of these two formulas shows that \( \chi(O_X(D)) \) equals \( 1 - g \).

Using Serre duality, we get the more classical form of Riemann-Roch

**Corollary 1.7.2.** Let \( K \) be a canonical divisor, then

\[
h^0(O(D)) - h^0(O(K - D)) = \deg D + 1 - g
\]

In particular

\[
h^0(O(D)) \geq \deg D + 1 - g \quad (\text{Riemann’s inequality})
\]

The Riemann-Roch theorem is a fundamental tool in the study of curves. One consequence is that any compact complex curve carries a nonconstant meromorphic function. From this, it is not difficult to deduce the following important fact:
Theorem 1.7.3. Every compact Riemann surface is a nonsingular projective algebraic curve.

So from here on, we won’t make a distinction between these notions. We also make use of GAGA (Serre, Géométrie algébrique et géométrie analytique) to switch between algebraic and analytic viewpoints whenever convenient.

Theorem 1.7.4 (Chow’s theorem/GAGA).

(a) Meromorphic functions on a projective varieties are rational

(b) Holomorphic maps between projective varieties are regular.

(c) Submanifolds of nonsingular projective varieties are subvarieties.

(d) Holomorphic vector bundles on projective varieties are algebraic, and their cohomology groups can be computed algebraically i.e. as the cohomology of the corresponding sheaves on the Zariski topology

As an easy application of Riemann-Roch, let us classify curves of genus \( \leq 2 \), where by curve we mean a nonsingular projective curve below.

Lemma 1.7.5. The only curves of genus \( 0 \) is \( \mathbb{P}^1 \).

Proof. Let \( X \) be a curve of genus 0. Let \( p \in X \). By Riemann-Roch \( h^0(\mathcal{O}(p)) \geq 2 \). This implies that there exists a nonconstant meromorphic function \( f \) having a pole of order 1 at \( p \) and no other singularities. Viewing \( f \) as a holomorphic map \( X \to \mathbb{P}^1 \), we have \( f^{-1}(\infty) = p \) and the ramification index \( e_p = 1 \). This means that the degree of \( f \) is 1, and one can see that this must be an isomorphism. \( \square \)

From the proof, we obtain the following useful fact.

Corollary 1.7.6. If for some \( p \), \( h^0(\mathcal{O}(p)) > 1 \), then \( X \cong \mathbb{P}^1 \).

We saw earlier that if \( f(x) \) is a degree 4, polynomial then \( y^2 = f \) has genus 1. Such a curve is called elliptic.

Lemma 1.7.7. Conversely, any genus 1 curve can be realized as degree 2 cover of \( \mathbb{P}^1 \) branched at 4 points.

Proof. Let \( X \) be a curve of genus 1. Let \( p \in X \). By Riemann-Roch \( h^0(\mathcal{O}(np)) \geq n \) for \( n \geq 0 \). Thus we can find a nonconstant function \( f \in H^0(\mathcal{O}(2p)) \). Either \( f \) has a pole of order 1 or order 2 at \( p \). The first case is ruled out by the last corollary, so we are in the second case. Viewing \( f \) as a holomorphic map \( X \to \mathbb{P}^1 \), we have \( f^{-1}(\infty) = p \) with \( e_p = 2 \). Therefore \( f \) has degree 2. Now apply the Riemann-Hurwitz formula to conclude that \( f \) has 4 branch points. \( \square \)

Lemma 1.7.8. Any genus 2 curve can be realized as degree 2 cover of \( \mathbb{P}^1 \) branched at 6 points.
Proof. Either using Riemann-Roch or directly from corollary 1.5.5, we can see that \( h^0(\Omega^1_X) = 2 \). Choose two linearly independent holomorphic 1-forms \( \omega_i \). The ratio \( f = \omega_2/\omega_1 \) is a nonconstant meromorphic function. Since \( \deg \text{div}(\omega_2) = 2(2) - 2 = 2 \), \( f \) has most two poles counted with multiplicity. Therefore \( f : X \to \mathbb{P}^1 \) has degree at most 2, and as above we can see that it is exactly 2. Again using the Riemann-Hurwitz formula shows that \( f \) has 6 branch points. \qed

1.8 Genus 3 curves

Continuing the analysis, we come to genus 3 curves. Here things become more complicated. Some of these curves are hyperelliptic, but not all.

Theorem 1.8.1. A genus 3 curve is either a hyperelliptic curve branched at 8 points or a nonsingular quartic in \( \mathbb{P}^2 \). These two cases are mutually exclusive.

Fix a genus 3 curve \( X \). Then \( h^0(\Omega^1_X) = 3 \). We can choose a basis \( \omega_0, \omega_1, \omega_2 \).

Lemma 1.8.2. For every \( p \in X \), one of the \( \omega_i(p) \neq 0 \). (This condition says that \( \Omega^1_X \) is generated by global sections, or in classical language that the canonical linear system is based point free.)

Proof. Suppose that all \( \omega_i(p) = 0 \) for some \( p \). Then \( \omega_i \) would define sections of \( \Omega^1_X(-p) \). So we must have \( h^0(\Omega^1_X(-p)) \geq 3 \). But by Riemann-Roch

\[
h^0(\Omega^1(-p)) - h^0(\mathcal{O}(p)) = 2(3) - 2 - 1 + (1 - 3) = 1
\]

Since \( h^0(\mathcal{O}(p)) = 1 \), we obtain \( h^0(\Omega^1(-p)) = 2 \). This is a contradiction. \( \square \)

Choosing a local coordinate \( z \), we can write \( \omega_i = f_i(z)dz \). By the previous lemma, the point \( [f_0(z), f_1(z), f_2(z)] \in \mathbb{P}^2 \) is defined at all points of the coordinate chart. If we change coordinates to \( \zeta \), then \( \omega_i = uf_i d\zeta \), where \( u = \frac{dz}{d\zeta} \). This means that the point \( [f_0(z), f_1(z), f_2(z)] \in \mathbb{P}^2 \) is globally well defined. In this way, we get a nonconstant holomorphic map

\[
\kappa : X \to \mathbb{P}^2
\]

called the canonical map. The image is a curve in \( \mathbb{P}^2 \), which we denote by \( Y \). We can factor \( \kappa = \pi : X \to Y \) followed by the inclusion \( Y \subset \mathbb{P}^2 \). Let \( d_1 \) be the degree of \( \pi \). Let \( x_i \) denote the homogenous coordinates of \( \mathbb{P}^2 \). Consider the line \( \ell \subset \mathbb{P}^2 \) defined by \( a_0x_0 + a_1x_1 + a_2x_2 = 0 \). If we assume that \( a_i \) are chosen generically, then \( \ell \cap Y \) is a union, of say \( d_2 = \deg Y \) “distinct points” (more precisely, the scheme theoretic intersection is a reduced subscheme of length \( d_2 \)). If we pull this back to \( X \) we get \( d_1d_2 \) points which coincides with the zero set of \( \sum a_i \omega_i \). The degree of a canonical divisor is 4. Thus we get three cases:

1. \( \deg Y = 4 \) and \( \pi \) has degree 1,
2. \( \deg Y = 2 \) and \( \pi \) has degree 2,
(3) $\deg Y = 1$ and $\pi$ has degree 4.

In fact, case (3) is impossible because it would imply a linear dependence between the $\omega_i$. In case (2), $Y$ is an irreducible conic, which is necessarily isomorphic to $\mathbb{P}^1$. So this is the hyperelliptic case. Using Hartshorne, chap IV, prop 5.3, we see that conversely a hyperelliptic genus 3 curve necessarily falls into case (2). By Riemann-Hurwitz this must be branched at 8 points.

To complete the analysis, let us take closer look at case (1). Here $\pi$ is birational. So $X$ is necessarily the normalization of $Y$.

**Lemma 1.8.3.** $Y$ is nonsingular. Therefore $X \to Y$ is an isomorphism.

**Proof.** Suppose that $Y$ is singular. Consider the sequence

$$0 \to \mathcal{O}_Y \to \pi_* \mathcal{O}_X \to C \to 0$$

where the cokernel $C$ is a sum of sky scraper sheaves

$$\bigoplus_p \left( \bigoplus_{q \in \pi^{-1}(p)} \mathcal{O}_{X,p} \right) / \mathcal{O}_{Y,p}$$

supported a the singular points of $Y$. Let $\delta_p$ denote the dimension of each summand above. At least one of these numbers is positive because $Y$ is singular. $C$ is flasque, therefore $H^1(Y, C) = 0$. Also since $\pi$ is finite, $H^i(X, \mathcal{O}_X) = H^i(Y, \pi_* \mathcal{O}_X)$ (c.f. Hartshorne, Algebraic Geometry, chap III ex 4.1). Consequently

$$\chi(\mathcal{O}_X) = \chi(\mathcal{O}_Y) + \sum \delta_p$$

and therefore

$$\chi(\mathcal{O}_X) > \chi(\mathcal{O}_Y)$$

The arithmetic genus $p_a$ of $Y$ is defined so $\chi(\mathcal{O}_Y) = 1 - p_a$. The inequality says that $p_a$ is strictly less that the genus of $X$. It is worth pointing that the argument is completely general; it applies to any singular curve. To obtain a contradiction, we show that $p_a = 3$. This will follow from the next result. 

**Theorem 1.8.4.** If $C \subset \mathbb{P}^2$ is a curve of degree $d$, then the arithmetic genus

($= \text{the ordinary genus when } C \text{ is nonsingular}$)

$$p_a = \frac{(d-1)(d-2)}{2}$$

Before starting the proof, we summarize some basic facts about projective space. Proofs of these statements can be found in Hartshorne. Let $x_0, \ldots, x_n$ be homogeneous coordinates of $\mathbb{P}_C^n$. We have an open cover consisting of $U_i = \{x_i \neq 0\}$. $U_i \cong \mathbb{A}^n$ with coordinates $\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i}$. Let $\mathcal{O}_{\mathbb{P}}(d)$ be the line bundle determined by the cocycle $\left( \frac{x_i}{x_j} \right)^d$. Suppose that $d \geq 0$. Given a homogeneous degree $d$ polynomial $f(x_0, \ldots, x_n)$, the transformation rule

$$f \left( \frac{x_0}{x_j}, \ldots, \frac{x_n}{x_j} \right) = \left( \frac{x_i}{x_j} \right)^d f \left( \frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i} \right)$$
shows that it determines a section of $H^0(\mathbb{P}^n, O_{\mathbb{P}^n}(d))$. In fact, all sections are of this form. Therefore
\[ h^0(O_{\mathbb{P}^n}(d)) = \binom{n + d}{n}, \quad d \geq 0 \]
All the other cohomology groups are zero. We also have a duality
\[ h^i(O_{\mathbb{P}^n}(-d)) = h^{n-i}(O_{\mathbb{P}^n}(-n - 1 + d)) \]
Therefore
\[ \chi(O_{\mathbb{P}^n}(-d)) = h^2(O(-d)) = \frac{(d - 1)(d - 2)}{2} \quad (1.5) \]
Under this identification
\[ H^0(\mathbb{P}^n, O_{\mathbb{P}^n}(d)) \cong H^0(\mathbb{P}^n, O_{\mathbb{P}^n}(d)) \]
a nonzero polynomial $f$ determines an injective morphism $O_{\mathbb{P}^n} \to O_{\mathbb{P}^n}(d)$. Tensoring with $O_{\mathbb{P}^n}(-d)$ yields an injective morphism $O_{\mathbb{P}^n}(-d) \to O_{\mathbb{P}^n}$. The image can be identified with the ideal sheaf generated by $f$.

**Proof of theorem.** We have an exact sequence
\[ 0 \to O_{\mathbb{P}^n}(-d) \to O_{\mathbb{P}^n} \to O_{\mathcal{C}} \to 0 \]
Therefore
\[ \chi(O_{\mathcal{C}}) = \chi(O_{\mathbb{P}^n}) - \chi(O_{\mathbb{P}^n}(-d)) \]
\[ = 1 - \frac{(d - 1)(d - 2)}{2} \]
All of these results taken together implies theorem 1.8.1.

### 1.9 Automorphic forms

We give one more application of the Riemann-Roch. The group $SL_2(\mathbb{R})$ acts on the upper half plane $\mathbb{H} = \{ z \in \mathbb{C} \mid \text{Im} z > 0 \}$ by fractional linear transformations. Suppose that $\Gamma \subset SL_2(\mathbb{R})$ is a discrete subgroup which acts freely on $\mathbb{H}$ (or more precisely suppose that $\Gamma/\{\pm I\}$ acts freely). Then the quotient $X = \mathbb{H}/\Gamma$ is naturally a Riemann surface. Let us assume that $X$ is compact. The genus $g$ can be shown to be at least two by the Gauss-Bonnet theorem. An automorphic form of weight $2k$ is a holomorphic function $f(z)$ on $\mathbb{H}$ satisfying
\[ f(z) = (cz + d)^{-2k} f \left( \frac{az + b}{cz + d} \right) \]
for each
\[ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \]
When $f(z)$ has weight 2, $f(z)dz$ is invariant under the group. Therefore it defines a holomorphic form on $X$. It follows that
Proposition 1.9.1. The dimension of the space of weight two automorphic forms is $g$.

If $f(z)$ has weight $2k$, then $f(z)dz^{\otimes k}$ defines a section of $(\Omega^1_X)^{\otimes k} = \mathcal{O}_X(kK)$. From Riemann-Roch

$$h^0(\mathcal{O}(kK)) - h^0(\mathcal{O}((1-k)K)) = \text{deg}(kK) + (1 - g) = (2k - 1)(g - 1)$$

If $k > 1$, then $\text{deg}(1-k)K < 0$. Therefore:

Proposition 1.9.2. If $k > 1$, the dimension of the space of weight $2k$ automorphic forms is $(g - 1)(2k - 1)$. 
Chapter 2

Divisors on a surface

2.1 Bezout’s theorem

Given distinct irreducible curves $C, D \subset \mathbb{P}^2$, $C \cap D$ is finite. The naive guess is that the number of points is the product of the degrees of (the defining equations of) $C$ and $D$. This is not literally true, unless the points are counted with multiplicity. Let us start by making this precise. Our goal is to define a local intersection number $(C \cdot D)_p$ for each $p \in C \cap D$, such that it equals 1 if $C$ and $D$ are smooth and transverse at $p$. We could try to count the number of points in neighbourhood of $p$ after perturbing the curves slightly, but it’s more efficient to define it algebraically. Consider the local ring $\mathcal{O}_p = \mathcal{O}_{\mathbb{P}^2, p}$, given by the stalk of the sheaf of regular functions. As usual $m$ denotes the maximal ideal. Then the ideals of $C$ and $D$ are defined by elements $f, g \in \mathcal{O}_p$. We define

$$(C \cdot D)_p = \dim \mathcal{O}_p/(f, g)$$

Note that this number is finite provided that $(f, g)$ is an $m$-primary ideal, and this holds in the present case. Had we used the stalk $\mathcal{O}_p^{an}$ of the sheaf of holomorphic functions in the above definition, we would have gotten the same result.

**Lemma 2.1.1.** $(C \cdot D)_p = \dim \mathcal{O}_p^{an}/(f, g) = \dim \hat{\mathcal{O}}_p/(f, g)$

**Proof.** Since $I = (f, g)$ is $m$-primary $\mathcal{O}_p/I \cong (\mathcal{O}_p/m^N)/I \cong (\hat{\mathcal{O}}_p/m^N)/I \cong \hat{\mathcal{O}}_p/m$. The same argument shows that $\mathcal{O}_p^{an}/I \cong \hat{\mathcal{O}}_p/m$. □

Let us check that it works as advertised

**Lemma 2.1.2.** If $C$ and $D$ are smooth and transverse at $p \in C \cap D$, then $(C \cdot D)_p = 1$.

**Proof.** We can identify the differentials $df$ and $dg$ with the images of $f$ and $g$ in $m/m^2$. The assumptions imply that these are linearly independent. Therefore $(f, g)$ generate $m/m^2$. Thus, by Nakayama, they generate $m$, so $\mathcal{O}_p/(f, g) = \mathbb{C}$. □
Theorem 2.1.3 (Bezout’s theorem). Let \(C\) and \(D\) be possibly reducible curves having no irreducible components in common, then

\[
\sum_{p \in C \cap D} (C \cdot D)_p = (\deg C)(\deg D)
\]

Although this is easy to prove by elementary methods, we will deduce it from something more general.

2.2 Divisors

By an algebraic surface \(X\), we will mean a two dimensional nonsingular projective variety over an algebraically closed field. We will work almost exclusively over \(\mathbb{C}\) for now. An irreducible closed curve \(C \subset X\) is also called a prime divisor. We allow \(C\) to be singular. Let \(\mathbb{C}(X)\) be the field of rational functions on \(X\). Let \(U \subset X\) is an affine open set with coordinate ring \(R\), then \(\mathbb{C}(X)\) is fraction field of \(R\). Let \(C \subset X\) be a prime divisor which meets \(U\). Then the ideal \(I_C\) of \(C \cap U\) is locally principle. This means that after shrinking \(U\), \(I_C = (f)\) for some nonunit \(f \in R\). The ideal \(I_C\) is a height one prime ideal, therefore the localization \(R_{I_C}\) is discrete valuation ring. Let \(\text{ord}_C : \mathbb{C}(X) \to \mathbb{Z} \cup \{\infty\}\) be the associated valuation. This depends only on \(C\) and not on \(R\). We can think of \(\text{ord}_C\) as the measure the order of zero or pole along \(C\).

Lemma 2.2.1. Given \(f \in \mathbb{C}(X)^*\), there exists only a finite number of irreducible curves for which \(\text{ord}_C(f) \neq 0\).

Proof. \(f\) is regular on some nonempty Zariski open set \(U \subset X\). The curves for which \(\text{ord}_C(f) < 0\) are irreducible components of \(X - U\), and there can be finitely many of these. The same argument applied to \(f^{-1}\) shows that there are finite many \(C\) with \(\text{ord}_C(f) > 0\).

A divisor is a finite integer linear combination \(D = \sum n_i C_i\) of prime divisors. As with curves, given \(f \in \mathbb{C}(X)^*\), we can associate a divisor

\[
\text{div}(f) = \sum_C \text{ord}_C(f)C \quad \text{(sum over irreducible curves)}
\]

Such a divisor is called principal. Since \(\text{div}\) is a homomorphism, the set of principal divisors forms a subgroup \(\text{Princ}(X)\) of all divisors \(\text{Div}(X)\). The quotient is the class group \(\text{Cl}(X)\). Two divisors are called linearly equivalent if they have the same image in \(\text{Cl}(X)\). We will use the symbol \(~\) for linear equivalence.

Locally a divisor is given by an equation \(f = 0\). Consequently, the ideal sheaf \(\mathcal{I}_C\) is locally principal, and therefore locally free of rank 1. We let \(\mathcal{O}_X(nC) = \mathcal{O}_X(\mathcal{I}_C^n)\), where the notation means tensor \(\mathcal{I}_C^{-1}\) with itself \(n\) times. We define

\[
\mathcal{O}_X(D) = \mathcal{O}_X(n_1 C_1) \otimes \mathcal{O}_X(n_2 C_2) \otimes \ldots
\]
This can also be identified with sheaf of fractional ideals
\[ \mathcal{O}_X(D)(U) = \{ f \in \mathcal{C}(X) \mid \forall C, C \cap U \neq 0 \Rightarrow \text{ord}_C(f) + D \geq 0 \} \]

The second description shows that a principal divisor gives a principal fractional ideal, which is isomorphic to \( \mathcal{O}_X \). There \( D \mapsto \mathcal{O}_X(D) \) gives a homomorphism from \( Cl(X) \) to the group of line bundle \( Pic(X) \). This is an isomorphism (Hartshorne, chap II, 6.16). A standard computation (loc. cit. 6.17) shows that \( Pic(\mathbb{P}^n) = \{ \mathcal{O}_{\mathbb{P}^n}(i) \} \cong \mathbb{Z} \).

### 2.3 Intersection Pairing

Let \( X \) be an algebraic surface. Given a line bundle \( \mathcal{L} \), let \( h^i(\mathcal{L}) = \dim H^i(X, \mathcal{L}) \). These are numbers are known to be finite, and zero when \( i > 2 \). Set

\[ \chi(\mathcal{L}) = h^0(\mathcal{L}) - h^1(\mathcal{L}) + h^2(\mathcal{L}) \]

**Theorem 2.3.1.** There exists a symmetric bilinear pairing

\[ Cl(X) \times Cl(X) \to \mathbb{Z} \]

such that

(a) \[ (C \cdot D) = \sum_{p \in C \cap D} (C \cdot D)_p \] (2.1)

whenever \( C \) and \( D \) are distinct irreducible curves.

(b) Let \( C, D \) be irreducible curves, and let \( \pi : \tilde{C} \to C \) be the normalization and \( \iota : \tilde{C} \to X \) the composition with the inclusion. Then

\[ C \cdot D = \deg(\iota^* \mathcal{O}_X(D)) \]

(c) \[ (C \cdot D) = \chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(-C)) - \chi(\mathcal{O}_X(-D)) + \chi(\mathcal{O}_X(-C - D)) \] (2.2)

**Proof.** We will use slick approach due to Mumford, Lectures on curves on an algebraic surface, pp 84-85. Define

\[ (C \cdot D) = \chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(-C)) - \chi(\mathcal{O}_X(-D)) + \chi(\mathcal{O}_X(-C - D)) \]

so that (c) holds by definiton. This gives a symmetric pairing on \( Cl(X) \cong Pic(X) \). The bilinearity will be proven later. If \( C \) and \( D \) are distinct irreducible curves, we have a Koszul resolution

\[ 0 \to \mathcal{O}_X(-C - D) \to \mathcal{O}_X(-C) \oplus \mathcal{O}_X(-C) \to \mathcal{O}_X \to \mathcal{O}_{C \cap D} \to 0 \]
where \( C \cap D \) refers to the scheme defined by \( \mathcal{I}_C + \mathcal{I}_D \). \( \mathcal{O}_{C \cap D} \) is a sum of skyscraper sheaves of length \( (C \cdot D)_p \) at each \( p \in C \cap D \). Hence \( h^0(\mathcal{O}_{C \cap D}) \) is the sum on the right of (2.1), and there are no higher cohomologies. Therefore

\[
\sum_{p \in C \cap D} (C \cdot D)_p = \chi(\mathcal{O}_{C \cap D})
\]

\[
= \chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(-C)) - \chi(\mathcal{O}_X(-D)) + \chi(\mathcal{O}_X(-C - D))
\]

This proves (2.1).

We will write \( \mathcal{O}_C(\pm D) \) instead of \( \iota^* \mathcal{O}_X(\pm D) \) below. We have exact sequences

\[
0 \to \mathcal{O}_X(-C) \to \mathcal{O}_X \to \mathcal{O}_C \to 0
\]

\[
0 \to \mathcal{O}_C \to \pi_* \mathcal{O}_\tilde{C} \to \pi_* \mathcal{O}_\tilde{C}/\mathcal{O}_C \to 0
\]

Tensor both with \( \mathcal{O}_X(-D) \) to obtain

\[
0 \to \mathcal{O}_X(-C - D) \to \mathcal{O}_X(-D) \to \mathcal{O}_C(-D) \to 0
\]

and

\[
0 \to \mathcal{O}_C(-D) \to \pi_* (\mathcal{O}_\tilde{C} \otimes \mathcal{O}_X(-D)) \to (\pi_* \mathcal{O}_\tilde{C}/\mathcal{O}_C) \otimes \mathcal{O}_X(-D) \to 0
\]

We can simplify the second sequence to

\[
0 \to \mathcal{O}_C(-D) \to \pi_* (\pi^* \mathcal{O}_C(-D)) \to \pi_* \mathcal{O}_\tilde{C}/\mathcal{O}_C \to 0
\]

using the projection formula

\[
\pi_* \pi^* (\mathcal{O}_C(-D)) \cong \pi_* (\mathcal{O}_\tilde{C})(-D)
\]

and the fact that \( \pi_* \mathcal{O}_\tilde{C}/\mathcal{O}_C \) is a sum of skyscraper sheaves. Using these sequences, we obtain

\[
\chi(\mathcal{O}_\tilde{C}) - \chi(\pi_* \mathcal{O}_\tilde{C}/\mathcal{O}_C) = \chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(-C))
\]

\[
\chi(\pi^* \mathcal{O}_C(-D)) - \chi(\pi_* \mathcal{O}_\tilde{C}/\mathcal{O}_C) = \chi(\mathcal{O}_X(-D)) - \chi(\mathcal{O}_X(-C - D))
\]

Subtract these and apply (2.2) to obtain

\[
C \cdot D = \chi(\mathcal{O}_\tilde{C}) - \chi(\pi^* \mathcal{O}_C(-D))
\]

Now use Riemann-Roch. This proves (b).

(b) implies additivity when one of the divisors is prime. For the general case, we need the following fact:

**Lemma 2.3.2** (Moving lemma). Any divisor \( D \) is linearly equivalent to a difference \( E - F \), where \( E \) and \( F \) are smooth curves in general position.
A proof of the lemma can be found on pp 358-359 of Hartshorne, although this isn’t stated as a lemma there. Let
\[ S(C_1, C_2, C_3) = (C_1 \cdot C_2 + C_3) - (C_1 \cdot C_2) - (C_1 \cdot C_3) \]
We need to show that \( s \) is identically zero. This is zero if \( C_1 \) is prime as noted above. The expression \( s \) is easily verified to be symmetric in \( C_1, C_2, C_2 \). Therefore it is zero if any of the \( C_i \) are prime.

Given divisors \( C \) and \( D \), apply the lemma to write \( D \sim E - F \) as above. Since \( s(C, D, F) = 0 \), we obtain
\[ (C \cdot D) = (C \cdot E) - (C \cdot F) \]
By what we said earlier, the right side is additive in first variable.

\[ \square \]

**Corollary 2.3.3.** *Bezout’s theorem holds.*

**Proof.** We give two proofs.

First using the fact that \( Pic(\mathbb{P}^2) = \mathbb{Z} \) with a generator corresponding to a line. Clearly, \( L \cdot L' = 1 \) for 2 distinct lines. This together with the other properties forces \( C \cdot D = \deg C \deg D \).

For the second proof, combine formulas (2.2) and (1.5) and simplify to obtain \( C \cdot D = \deg C \deg D \).

\[ \square \]

We can give the intersection pairing a topological interpretation. We can define the first Chern class
\[ c_1 : Pic(X) \to H^2(X, \mathbb{Z}) \]
as before as the connecting map associated to the exponential sequence. Given an irreducible curve \( C \subset X \), \( H_2(C, \mathbb{Z}) = \mathbb{Z} \) with a preferred generator determined by the orientation coming from the complex structure. Let \( [C] \in H_2(X, \mathbb{Z}) \) be the push forward of this class.

**Theorem 2.3.4.** *Given irreducible curves \( C, D \subset X \)
\[ C \cdot D = \langle c_1(\mathcal{O}_X(D)), [C] \rangle \]
where \( \langle \cdot, \cdot \rangle \) here denotes evaluation of a cohomology class on a homology class.*

**Proof.** Let \( \iota : \tilde{C} \to X \) be the composition of the normalization \( \tilde{C} \to C \) and the inclusion. If \( [\tilde{C}] \in H_2(\tilde{C}, \mathbb{Z}) \) is the preferred generator, then \( \iota_*[\tilde{C}] = [C] \). Then
\[ \langle c_1(\mathcal{O}_X(D)), [C] \rangle = \langle \iota^*\mathcal{O}_X(D), [\tilde{C}] \rangle = \deg \mathcal{O}_C(D) \]
Now apply (b) of the previous theorem.

\[ \square \]
This can be put in more symmetric form with the help of Poincaré duality, which gives an isomorphism
\[ H_2(X, \mathbb{Z}) \cong H^2(X, \mathbb{Z}) \]

Under this isomorphism one has that \([C]\) corresponds to \(c_1(O(C))\). There is a natural pairing
\[ H^2(X, \mathbb{Z}) \otimes H^2(X, \mathbb{Z}) \to \mathbb{Z} \]
given by cup product followed by evaluation to the fundamental class. Then we have

**Theorem 2.3.5.** \( C \cdot D = c_1(O_X(C)) \cdot c_1(O_X(D)) \)

Let \( f : X \to Y \) be a surjective morphism of smooth projective varieties of the same dimension \( n \). Then we get a finite extension of function field \( \mathbb{C}(Y) \subset \mathbb{C}(X) \). Its degree is (by definition) the degree of \( f \). Let’s call this \( d \). We can understand this more geometrically as follows. After restricting to a Zariski open set \( U \subset Y \), \( f^{-1}U \to U \) becomes étale. Topologically, this is a covering space, and \( d \) is the number of sheets. Given a divisor \( D \subset Y \), locally given by an equation \( g = 0 \). \( f^*D \) is the locally defined by \( f^*g = 0 \). Note that even if \( D \) is prime, \( f^*D \) need not be. This operation is compatible with pullback of line bundles.

**Corollary 2.3.6.** \( f : X \to Y \) be a surjective morphism smooth projective surfaces. Then \( f^*C \cdot f^*D = (\deg f)(C \cdot D) \)

**Proof.** We have
\[
f^*C \cdot f^*D = \int_X f^*[c_1(O_X(C)) \cup c_1(O_X(D))] = \deg f \int_Y c_1(O_X(C)) \cup c_1(O_X(D))
\]

\[ \square \]

### 2.4 Adjunction formula

Given a surface \( X \), a two form \( \alpha \) is given locally by \( f(x, y)dx \wedge dy \), where \( x, y \) are coordinates. Given a prime divisor \( C \), the number \( \text{ord}_C(\alpha) = \text{ord}_C(f) \) is well defined. Therefore we can define a divisor
\[
\text{div}(\alpha) = \sum_C \text{ord}_C(\alpha)C
\]

The class in \( Cl(X) \) is well defined and referred to as the canonical divisor (class) \( K = K_X \).

**Theorem 2.4.1** (Adjunction formula). Let \( C \subset X \) be a smooth curve of genus \( g \), then
\[
2g - 2 = (K + C) \cdot C
\]
The proof will be broken down into a couple of lemmas. Fix \( \iota : C \to X \) as above. Given a vector bundle (= locally free sheaf) \( V \) of rank \( r \), let \( \det V = \wedge^r V \). This is a line bundle.

**Lemma 2.4.2.** Given an exact sequence of vector bundles

\[
0 \to V \to W \to U \to 0
\]

\[
\det W \cong \det V \otimes \det U
\]

**Proof.** Exactly as for line bundles, a rank \( r \) vector bundle is determined by an open covering \( \{ U_i \} \) and a collection of holomorphic maps \( g_{ij} : U_{ij} \to GL_r(\mathbb{C}) \) satisfying the cocycle identity. This is constructed from a local trivialization. Choose compatible local trivializations for \( V \) and \( W \). Then the cocycle for \( W \) is of the form

\[
g_{ij} = \begin{pmatrix} h_{ij} & \ast \\ 0 & \ell_{ij} \end{pmatrix}
\]

where \( h_{ij}, \ell_{ij} \) are cocycles for \( V \) and \( U \). This implies that

\[
\det g_{ij} = \det h_{ij} \det \ell_{ij}
\]

and therefore that \( \det W \cong \det V \otimes \det U \).

The theorem will follow from theorem 2.3.1(b) and the next result (which also called the adjunction formula).

**Lemma 2.4.3.** \( \Omega_C^1 \cong \Omega_X^2|_C \otimes \mathcal{O}_C(C) \) where \( \Omega_X^2|_C = \iota^* \Omega_X^2 \).

**Proof.** We have an exact sequence of locally free sheaves

\[
0 \to \mathcal{O}_C(-C) \to \Omega_X^1|_C \to \Omega_C^1 \to 0
\]

By the previous lemma, we obtain

\[
\Omega_C^1 \otimes \mathcal{O}_C(-C) \cong \Omega_X^2|_C
\]

Tensor both sides by \( \mathcal{O}_C(C) \) to obtain the lemma.

The adjunction formula gives a very efficient method for computing the genus. As a first application, let us reprove theorem 1.8.4 for smooth curves. First we observe that:

**Lemma 2.4.4.** The canonical bundle of projective space is

\[
\Omega_{\mathbb{P}^2}^2 \cong \mathcal{O}_{\mathbb{P}^2}(-3)
\]

**Proof.** Let \( x_0, x_1, x_2 \) homogeneous coordinates, then the ratios \( x_i/x_j \) give coordinates on the patch \( U_j = \{ x_j \neq 0 \} \). A direct computation gives

\[
d \left( \frac{x_1}{x_0} \right) \wedge d \left( \frac{x_2}{x_0} \right) = -\left( \frac{x_0}{x_1} \right)^{-3} d \left( \frac{x_0}{x_1} \right) \wedge d \left( \frac{x_2}{x_1} \right)
\]

which shows that the form on the left has singularity of order 3 on the line \( x_0 = 0 \).
Remark 2.4.5. More generally, we have

\[ \Omega^n_{\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}^n}(-n-1) \]

Example 2.4.6. Suppose that \( C \subset \mathbb{P}^2 \) is a smooth curve of degree \( d \), then

\[ 2g - 2 = (K + C) \cdot C = (-3 + d)(d) \]

solving for \( g \) gives us the earlier formula

\[ g = \frac{(d-1)(d-2)}{2} \]

Example 2.4.7. Let \( X = \mathbb{P}^1 \times \mathbb{P}^1 \). We can see from Künneth’s formula that \( H^2(X, \mathbb{Z}) = \mathbb{Z}^2 \), with two generators corresponding to the lines \( L = \mathbb{P}^1 \times \{0\} \) and \( M = \{0\} \times \mathbb{P}^1 \). It follows from theorem 2.3.1 that \( L^2 = M^2 = 0 \) and \( L \cdot M = 1 \). Let \( K = aL + bM \) be the canonical divisor. By the adjunction formula,

\[ -2 = L \cdot (K + L) = b \]

\[ -2 = M \cdot (K + M) = a \]

A curve \( C \) defined by a bihomogeneous polynomial of bidegree \((d,e)\), can be seen to be linearly equivalent to \( dL + eM \). The genus is given by

\[ 2g - 2 = C \cdot (K + C) = (dL + eM)((d - 2)L + (e - 2)M) = d(e - 2) + e(d - 2) \]

2.5 Riemann-Roch

Let \( X \) be a smooth projective surface as before.

Theorem 2.5.1 (Riemann-Roch for Surfaces). Given a divisor

\[ \chi(\mathcal{O}_X(D)) = \frac{1}{2}D \cdot (D - K) + \chi(\mathcal{O}_X) \]

As in the curve case, the key ingredient is Serre duality.

Theorem 2.5.2 (Serre duality). Given an \( n \) dimensional compact complex manifold \( X \), there is an isomorphism

\[ H^i(X, \mathcal{L}) \cong H^{n-i}(X, \omega_X \otimes \mathcal{L}^{-1})^* \]

where \( \omega_X = \Omega^n_X \) is the canonical sheaf.

Proof of theorem 2.5.1. Using the formula in (2.2), we obtain

\[ (-D) \cdot (D - K) = \chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(D)) - \chi(\mathcal{O}_X(K - D)) + \chi(\mathcal{O}_X(K)) \]

Note that \( \omega_X = \mathcal{O}_X(K) \), so by Serre duality, \( \chi(\mathcal{O}_X) = \chi(\mathcal{O}_X(K)) \) and \( \chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X(K - D)) \). Therefore

\[ -D \cdot (D - K) = 2(\chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(D))) \]

and the theorem follows.
Corollary 2.5.3. \( h^0(\mathcal{O}(D)) + h^0(\mathcal{O}(K - D)) \geq \frac{1}{2} D \cdot (D - K) + \chi(\mathcal{O}_X) \)

In order to sharpen the above inequality we need to eliminate the second term on the left. Here is a simple way to do it.

**Lemma 2.5.4.** If some positive multiple of a divisor \( E \) is linearly equivalent to an nonzero effective divisor, then \( H^0(\mathcal{O}(-E)) = 0 \).

**Proof.** Suppose that \( nE \) is equivalent to \( E' \) effective for some \( n > 0 \). Embed \( X \subset \mathbb{P}^N \), and let \( C = X \cap H \) for some general hyperplane. By Bertini, \( C \) is nonsingular. Then \( \deg \mathcal{O}(E)|_C = \frac{1}{n}(C \cdot E') > 0 \) because \( H \) must meet \( E' \). If \( f \in H^0(X, \mathcal{O}(-E)) \) is nonzero, then \( f \) restricts to a nonzero section of \( \mathcal{O}(-E)|_C \). But this impossible, since the degree is negative. Therefore \( H^0(X, \mathcal{O}(-E)) = 0 \).

**Corollary 2.5.5.** If \( D - K \) is linearly equivalent to an effective divisor then \( h^0(\mathcal{O}(D)) \geq \frac{1}{2} D \cdot (D - K) + \chi(\mathcal{O}_X) \)

**Proof.** If \( D - K \) is linearly equivalent to 0, then the corollary reduces to equality \( \chi(\mathcal{O}(K)) = \chi(\mathcal{O}_X) \) noted earlier. Otherwise apply the lemma to obtain \( H^0(X, \mathcal{O}(K - D)) = 0 \). The together with the previous corollary implies the result.

When Riemann-Roch is combined with Kodaira’s vanishing theorem or one of its refinements, we can get an exact formula for the dimension of global sections. Kodaira’s theorem is discussed in Griffiths-Harris. For the more recent results, see Lazarsfeld, Positivity in Algebraic Geometry.

**Theorem 2.5.6 (Kawamata-Viehweg/Kodaira).** If \( D \) is a divisor such that \( D^2 > 0 \) and \( D \cdot C \geq 0 \) for every effective divisor \( C \),

\[
H^i(X, \mathcal{O}_X(K + D)) = 0, \quad i > 0
\]

In particular, this holds when \( D \) is ample i.e. a positive multiple is equivalent to the pullback of a hyperplane for some projective embedding of \( X \).

**Corollary 2.5.7.** If \( D \) is as above,

\[
h^0(\mathcal{O}(K + D)) = \frac{1}{2} D(K + D) + \chi(\mathcal{O}_X)
\]

The Riemann-Roch theorem as stated above, gives no information about \( \chi(\mathcal{O}_X) \). In fact there is a more precise form of Riemann-Roch due to Hirzebruch which yields

\[
\chi(\mathcal{O}_X) = \frac{K^2 + e(X)}{12}
\]

where the expressions are the self intersections of the canonical divisor and the topological Euler characteristic. In fact an equivalent statement was obtained by Max Noether over 100 years ago, consequently this often refered to as “Noether’s formula”.

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Now we give an application automorphic forms in two variables. Let \( \Gamma \subset SL_2(\mathbb{R}) \times SL_2(\mathbb{R}) \) be discrete subgroup. It acts on the product of two copies of the upper half plane \( \mathbb{H}^2 \). We assume that the action is fixed point free and that the quotient \( X = \mathbb{H}^2/\Gamma \) is compact. This is a complex manifold. In fact, using Kodaira’s embedding theorem (see Griffiths-Harris) we can see that \( X \) is a smooth projective surface and \( K \) is ample. A section of \( \mathcal{O}(mK) \) can be interpreted as an automorphic form of weight \( 2m \), i.e. a holomorphic function on \( \mathbb{H}^2 \) satisfying

\[
f \left( \frac{a_1z_1 + b_1}{c_1z_1 + d_1}, \frac{a_2z_2 + b_2}{c_2z_2 + d_2} \right) = (c_1z_1 + b_1)^{2m}(c_2z_2 + b_2)^{2m}f(z_1, z_2)
\]

for every element of \( \Gamma \). From Riemann-Roch together with Kodaira vanishing, we obtain

\[
\text{Dim of wt } 2m \text{ automorphic forms } = \frac{1}{2}m(m - 1)K^2 + \chi(\mathcal{O}_X)
\]

### 2.6 Blow ups and Castelnuovo’s theorem

The blow up of the plane at the origin

\[
Bl_0 \mathbb{A}^2 = \{(x, \ell) \in \mathbb{A}^2 \times \mathbb{P}^1 \mid x \in \ell \}
\]

This is easily seen to be a quasiprojective variety. This comes with a map \( \pi : Bl_0 \mathbb{A}^2 \to \mathbb{A}^2 \) given by projection. It is an isomorphism over \( \mathbb{A}^2 - \{0\} \). The fibre \( \pi^{-1}0 \cong \mathbb{P}^1 \) is called the exceptional divisor. More generally, one can blow a point on any smooth projective surface \( X \) to get a new smooth projective surface \( Bl_pX \) with a map \( \pi : Bl_pX \to X \) which is an isomorphism over \( X - p \) and with \( \pi^{-1}p = E \cong \mathbb{P}^1 \). Let us denote \( Bl_pX \) by \( Y \).

**Proposition 2.6.1.**

(a) \( \pi^*C \cdot \pi^*D = C \cdot D \)

(b) \( \pi^*C \cdot E = 0 \)

(c) \( E^2 = -1 \)

**Proof.** Then \( \pi : Y \to X \) has degree 1, therefore by corollary 2.3.6 implies (a). By lemma 2.3.2, we can assume that \( p \notin C \). Therefore \( \pi^*C \cdot E = 0 \) because they are disjoint. For (c), choose a smooth curve \( C \) containing \( p \). The set theoretic preimage \( \pi^{-1}C \) is a union of \( E \) and a curve \( C' = C - \{p\} \) called the strict transform. This means that \( \pi^*C \) is a linear combination of these two divisors. To calculate the coefficients, we can choose local analytic coordinates \( x, y \) about \( p \) so that \( x \) gives the local equation of \( C \). On \( Y \) we have a chart with coordinates \( y, t = x/y \). Then \( \pi^*C \) is defined by \( yt = x = 0 \). So locally it factors into \( y = 0 \) and \( t = 0 \) which define \( E \) and \( C \) respectively. This means that \( \pi^*C = C' + E \).
Theorem 2.6.2 (Castelnuovo). If $Y$ is a surface with an curve $E \cong \mathbb{P}^1$ with $E^2 = -1$, then it must be the blow up of a smooth surface $X$ such that $E$ is the exceptional divisor.

Sketch. We outline the main step of the proof. First let us indicate the broad strategy. If $\pi : X \to Y$ exists and because our surfaces are projective, we have an embedding $\iota : Y \subset \mathbb{P}^N$. The restriction $\mathcal{O}_Y(1) = \iota^*\mathcal{O}_{\mathbb{P}^N}(1)$ gives a line bundle which is generated by global sections say $f_0, \ldots, f_N$. We can pull the bundle and it sections back to $X$ to get $\pi^*f_i$. Since these are locally just functions, it makes sense to map $x \mapsto [\pi^*f_0(x), \ldots, \pi^*f_N(x)] \in \mathbb{P}^N$. In this way, recover $Y$ as the image of this map in $\mathbb{P}^N$. The trick is to do this without knowing $Y$. Let us start with a very ample divisor $H$ on $X$. This means that $H$ is the restriction of hyperplane to $X$ under an embedding $X \subset \mathbb{P}^M$. It is technically can convenient that $H^1(X, \mathcal{O}(H)) = 0$. This can always be achieved by replacing $H$ by a positive multiple, if necessary, by a theorem of Serre. Now let $D = H + nE$, where $n = E \cdot H$. Observe that $D \cdot E = 0$.

We claim that $H^1(X, \mathcal{O}_X(H + iE)) = 0$ for $i = 0, \ldots, n$. For $i = 0$, this is by assumption. Suppose it’s true for $i < n$. Then from the sequence

$$0 \to \mathcal{O}_X(H + iE) \to \mathcal{O}_X(H + (i + 1)E) \to \mathcal{O}_E(H + (i + 1)E) \to 0 \quad (2.3)$$

Observe that $E = \mathbb{P}^1$ and $D \cdot E = 0$. Therefore

$$\mathcal{O}_E(H + (i + 1)E) \cong \mathcal{O}_{\mathbb{P}^1}(n - (i + 1))$$

So we get

$$H^1(X, \mathcal{O}_X(H + iE)) \to H^1(X, \mathcal{O}(H + (i + 1)E)) \to H^1(\mathbb{P}^1, \mathcal{O}(d))$$

where $d = n - (i + 1) \geq 0$. By induction, $H^1(X, \mathcal{O}_X(H + iE)) = 0$. By Serre duality

$$H^1(\mathbb{P}^1, \mathcal{O}(d)) \cong H^0(\mathbb{P}^1, \mathcal{O}(-2 - d)) = 0$$

This implies the claim.

Next, we claim that $\mathcal{O}_X(D)$ is generated by global sections. This means that for any $p \in X$, we can find global section which is nonzero at $p$. For $p \notin E$ this is easy. Since $H$ is very ample, $\mathcal{O}_X(H)$ is generated by generated by global sections. Under the inclusion $\mathcal{O}_X(H) \subset \mathcal{O}_X(D)$, these give sections $\mathcal{O}(D)$. For any $p \notin E$, we can find such a section nonzero at $p$. From (2.3), we obtain

$$H^0(\mathcal{O}_X(D)) \to H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \to H^1(\mathcal{O}_X(H + (n - 1)E)) = 0$$

This implies that we can lift the constant section 1 on $E$ can be lifted to a global section.
Now we can map $\pi : X \to \mathbb{P}^N$, where $N = \dim H^0(X, \mathcal{O}_X(D))$, using global sections as above. Along $E$, these sections are all constant. Therefore $\pi(E)$ is a point. Let $Y = \pi(X)$. This will turn out to be the desired variety. It remains to check that $Y$ is smooth, and that $X$ is the blow up of $Y$ at the image $\pi(E)$. For these details, we refer to Hartshorne, pp 415-416.
Chapter 3

Abelian varieties

3.1 Elliptic curves

An elliptic curve is a curve $X$ of genus one with a distinguished point 0. Topologically it is looks like a torus. A basic example is given as follows. A subgroup $L \subset \mathbb{C}$ generated by a real basis of $\mathbb{C}$ is called a lattice. As an abstract group $L \cong \mathbb{Z}^2$. The group quotient $\mathbb{C}/L$ has the structure of genus one curve with the image of 0 as the distinguished point. This has a commutative group law such that the group operations are holomorphic. Therefore these operations are regular by GAGA. In fact, one can see this more directly by embedding $\mathbb{C}/L \subset \mathbb{P}^2$ as a cubic so that 0 maps to an inflection point. Then $p + q + r = 0$ precisely when $p, q, r$ are collinear. The details can be found in any book on elliptic curves such as Silverman.

We want to show that all examples of elliptic curves are given as above.

**Theorem 3.1.1.** Any elliptic curve is isomorphic to $\mathbb{C}/L$, for some lattice.

There are a number of ways to prove this. We use an argument which will generalize. We have that $H^0(X, \Omega^1_X) = 1$. Pick a nonzero element $\omega$ in it. Also choose a basis $\alpha, \beta$ for $H^1(X, \mathbb{Z}) \cong \mathbb{Z}^2$.

**Proposition 3.1.2.** The integrals $\int_\alpha \omega, \int_\beta \omega$ are linearly independent over $\mathbb{R}$. Therefore they generate a lattice $L \subset \mathbb{C}$ called the period lattice.

**Proof.** We first redefine $L$ in a basis free way. We have a map $I : H_1(X, \mathbb{Z}) \to H^0(X, \Omega^1_X)^*$ which sends $\gamma \mapsto \int_\gamma$, and $L$ is just the image. The proposition is equivalent to showing that the $\mathbb{R}$-linear extension

$$I_\mathbb{R} : H_1(X, \mathbb{Z}) \otimes \mathbb{R} \to H^0(X, \Omega^1_X)^*$$

is an injection of real vector spaces. It is enough to show that it is a surjection, because they have the same dimension. The dual map

$$I_\mathbb{R}^* : H^0(X, \Omega^1_X) \to \text{Hom}(H_1(X, \mathbb{Z}), \mathbb{R}) \cong H^1(X, \mathbb{R})$$
factors through the natural map

\[ H^0(X, \Omega_X^1) \to H^1(X, \mathbb{C}) \]

which, as we saw earlier, is an inclusion. This implies that \( I_R \) is surjective.

We define the Jacobian \( J(X) = \mathbb{C}/L \). Let \( \int_0^p \omega \) denote the integral with respect to some path connecting 0 to \( p \). Let \( AJ(p) \) denote the image in \( J(X) \). Since we are dividing out by periods, this is independent of the chosen path. The map \( AJ : X \to J(X) \) is called the Abel-Jacobi map.

**Proposition 3.1.3.** \( AJ \) is holomorphic.

**Proof.** Let \( \tilde{X} \) denote the universal cover of \( X \). This is naturally a complex manifold, and it is enough to show that the induced map \( \tilde{AJ} : \tilde{X} \to J(X) \) is holomorphic. Since \( \tilde{X} \) is simply connected, \( H^1(\tilde{X}, \mathbb{C}) = 0 \). Therefore the pullback of \( \omega \) to \( \tilde{X} \) equals \( df \) for some \( C^\infty \) function \( f \). This is holomorphic because \( \omega \) is. Therefore

\[ \tilde{AJ}(p) = \int_0^p df = f(p) - f(0) \]

is holomorphic. \( \square \)

**Proposition 3.1.4.** \( AJ \) is a holomorphic isomorphism.

**Proof.** Since we already know that \( AJ \) is holomorphic, it is enough to show that is a diffeomorphism. We saw that \( H^0(X, \Omega_X^1) \cong H^1(X, \mathbb{R}) \). Under this identification, we can describe \( AJ \) as given by integration of \( C^\infty \) closed 1-forms. Note any two 2-tori are diffeomorphic, so we can assume that \( X = \mathbb{R}^2/\mathbb{Z}^2 \). If \( x, y \) are standard coordinates on \( \mathbb{R}^2 \), then \( dx, dy \) give a basis for \( H^1(X, \mathbb{R}) \). We may write

\[ AJ(a, b) = (\int_{(0,0)}^{(a,b)} dx, \int_{(0,0)}^{(a,b)} dy) = (a, b) \mod \text{periods} \]

using the straight line path, and the period lattice can be identified with the standard lattice. \( \square \)

This concludes the proof of theorem 3.1.1.

**Corollary 3.1.5.** Any elliptic curve is isomorphic to \( \mathbb{C}/\mathbb{Z} + \mathbb{Z} \tau \), where \( \tau \in \mathbb{H} \).

**Proof.** We know that \( X = \mathbb{C}/\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \) with \( \omega_1/\omega_2 \notin \mathbb{R} \). Then \( \tau = (\omega_1/\omega_2)^{\pm 1} \in \mathbb{H} \) and multiplication by some constant induces an isomorphism \( X = \mathbb{C}/\mathbb{Z} + \mathbb{Z} \tau \). \( \square \)
One can show that
\[ \mathbb{C}/\mathbb{Z} + \mathbb{Z} \tau \cong \mathbb{C}/\mathbb{Z} + \mathbb{Z} \tau' \]
if and only if \( \tau \) and \( \tau' \) lie in the same orbit of \( SL_2(\mathbb{Z}) \) if and only if \( j(\tau) = j(\tau') \), where
\[ j(\tau) = 1728 \frac{g_2(\tau)^3}{g_2(\tau)^3 - 27g_3(\tau)^2} \]
with
\[ g_2(\tau) = 60 \sum_{(m,n)\in\mathbb{Z}^2-\{(0,0)\}} \frac{1}{(m\tau + n)^4} \]
\[ g_3(\tau) = 140 \sum_{(m,n)\in\mathbb{Z}^2-\{(0,0)\}} \frac{1}{(m\tau + n)^6} \]

### 3.2 Abelian varieties and theta functions

A complex torus is quotient \( V/L \) of a finite dimensional complex vector space \( V \cong \mathbb{C}^n \) by a lattice \( L \cong \mathbb{Z}^{2n} \). This is a complex Lie group, that it is a complex manifold with a group structure whose operations are holomorphic. We say that a complex torus is an abelian variety if it can be embedded into some complex projective space \( \mathbb{P}^N \) as a complex submanifold. GAGA would imply that it is a projective variety and the that the group operations are regular.

Our first goal is to find a more convenient criterion for a torus to be an abelian variety. Let us start with an elliptic curve \( X = \mathbb{C}/\mathbb{Z} + \mathbb{Z} \tau \). We know that \( X \) can be embedded into projective space in principle, but we would like to do this as explicitly as possible. So need to construct functions \( f_i : \mathbb{C} \to \mathbb{C} \) such that \( p \mapsto [f_0(p), \ldots, f_N(p)] \in \mathbb{P}^N \) is well defined and gives an embedding. In order for it to be well defined, we need “quasiperiodicity”
\[ f_i(p + \lambda) = \text{(some factor)} f_i(p), \quad \forall \lambda \in \mathbb{Z} + \mathbb{Z} \tau \]
where the factor in front is the same for all \( i \) and nonzero. To begin with, we construct the Jacobi \( \theta \)-function is given by the Fourier series
\[ \theta(z) = \sum_{n \in \mathbb{Z}} \exp(\pi i n^2 \tau + 2\pi i n z) = \sum_{n \in \mathbb{Z}} \exp(\pi i n^2 \tau) \exp(2\pi i n z) \]
Writing \( \tau = x + iy \), with \( y > 0 \), shows that on a compact subset of the \( z \)-plane the terms are bounded by \( O(e^{-n^2y}) \). So uniform convergence on compact sets is guaranteed. This is clearly periodic
\[ \theta(z + 1) = \theta(z) \quad \text{(3.1)} \]
In addition it satisfies the function equation
\[ \theta(z + \tau) = \sum_{n \in \mathbb{Z}} \exp(\pi i n^2 \tau + 2\pi i n (z + \tau)) = \sum_{n \in \mathbb{Z}} \exp(\pi i (n + 1)^2 \tau - \pi i \tau + 2\pi i n z) = \exp(-\pi i \tau - 2\pi i z) \theta(z) \quad \text{(3.2)} \]
It turns out that $\theta$ spans the space of solutions to these equations. We get more solutions by relaxing these conditions. Let $N > 0$ be an integer, and consider the space $V_N$ of holomorphic functions satisfying

$$
\begin{align}
 f(z + N) &= f(z) \\
 f(z + N\tau) &= \exp(-\pi i N^2 \tau - 2\pi i N z)f(z)
\end{align}
$$

Any function in $V_N$ can be expanded in a Fourier series by the first equation, and the second equation gives recurrence conditions for the coefficients. This leads to

**Lemma 3.2.1.** $\dim V_N = N^2$.

*Proof.* See Mumford’s Lectures on Theta 1, page 9, for details.

The functions

$$
\theta_{a,b}(z) = \exp(\pi i a^2 \tau + 2\pi i a(z + b))\theta(z + a\tau + b), \quad a, b \in \frac{1}{N}\{0, \ldots, N - 1\}
$$

lie in $V_N$ and are independent. Therefore they form a basis.

**Lemma 3.2.2.** Given nonzero $f \in V_N$, it has exactly $N^2$ zeros, counted with multiplicities, in the parallelogram with vertices $0, N, N\tau, N + \tau$ (where we translate if necessary so no zeros lie on the boundary).

*Proof.* Complex analysis tells us that the number of zeros is given by the integral

$$
\frac{1}{2\pi i} \int_{C_1 + C_2 + C_3 + C_4} \frac{f'(z)dz}{f(z)}
$$

over the boundary of the parallelogram.

Using $f(z + N) = f(z)$, we obtain

$$
\int_{C_1 + C_3} \frac{f'(z)dz}{f(z)} = 0
$$

and from $f(z + N\tau) = Const. \exp(-2\pi i N z)f(z)$, we obtain

$$
\int_{C_2 + C_4} \frac{f'(z)dz}{f(z)} = 2\pi i N^2
$$

\hfill \Box
Theorem 3.2.3. Choose an integer \( N > 1 \) and basis \( f_i \) of \( V_N \). The map of \( \phi: \mathbb{C}/L \rightarrow \mathbb{P}^{N^2-1} \) by \( z \mapsto [f_i(z)] \) is an embedding.

Proof. Suppose that \( \phi \) is not one to one. Say that \( f(z_1) = f(z'_1) \) for some \( z_1 \neq z'_1 \) in \( \mathbb{C}/L \) and all \( f \in V_N \). By translation by \( (a\tau + b)/N \) for \( a, b \in \frac{1}{N}\mathbb{Z} \), we can find another such pair \( z_2, z'_2 \) with this property. Choose additional points, so that \( z_1, \ldots, z_{N^2-1} \) are distinct in \( \mathbb{C}/NL \). We define a map \( V_N \rightarrow \mathbb{C}^{N^2-1} \) by \( f \mapsto (f(z_i)) \). Since \( \dim V_N = N^2 \), we can find a nonzero \( f \in V_N \) so that \( f(z_1) = f(z_2) = f(z_3) = \ldots f(z_{N^2-1}) = 0 \)

Notice that we are forced to also have \( f(z'_1) = f(z'_2) = 0 \) which means that \( f \) has at least \( N^2 + 1 \) zeros which contradicts the lemma.

A similar argument shows that the derivative \( d\phi \) is nonzero.

Now let us consider a general \( g \) dimensional complex torus \( X = \mathbb{C}^g/L \). In order to generalize the previous arguments, let us assume that the lattice is a special form. Suppose that \( \Omega \) is a \( g \times g \) symmetric matrix with positive definite imaginary part. This will play the role of \( \tau \). The set of such matrices is called the “Siegel upper half plane”, and we denote it by \( \mathbb{H}_g \). Suppose that \( L = \mathbb{Z}^g + \mathbb{Z}^g \Omega \) integer column space of \( \Omega \)

Then we now construct the Riemann theta function on \( \mathbb{C}^g \)

\[
\theta(z) = \sum_{n \in \mathbb{Z}^g} \exp(\pi i n^t \Omega n + 2\pi i n^t z)
\]

which generalizes Jacobi’s theta function. We note because of the assumptions, we made about \( \Omega \), this series will converge to a holomorphic function on \( \mathbb{C}^g \) which is quasiperiodic. With this function hand we can construct a large class of auxiliary functions \( \theta_{a,b} \) as before, and use these to construct an embedding of \( X \) into projective space.

Theorem 3.2.4 (Riemann). If \( \Omega \in \mathbb{H}_g \) then \( \mathbb{C}^g/\mathbb{Z}^g + \mathbb{Z}^g \Omega \) is an abelian variety.

We want to ultimately rephrase the condition in a coordinate free language. But first we have to do some matrix calculations. Let \( \Pi = (I, \Omega) \) and let

\[
E = \begin{pmatrix} 0 & I \\
-I & 0 \end{pmatrix}
\]

be \( 2g \times 2g \). \( E \) determines a symplectic pairing \( L \times L \rightarrow \mathbb{Z} \)

One can check that the following identity holds

\[
E(iu, iv) = E(u, v)
\]

(3.4)

Then \( \Omega \in \mathbb{H}_g \) is equivalent to
1. $\Pi E^{-1} \Pi^T = 0$ and 
2. $i \Pi E^{-1} \Pi^T$ is positive definite.

Set 
\[ H(u, v) = E(iu, v) + iE(u, v) \] (3.5)

Then the following is easy to check directly, or see Birkenhake-Lange §4.2.

**Lemma 3.2.5.** The above conditions are equivalent to $H$ being positive definite hermitian.

An integer valued nondegenerate symplectic pairing $E$ on $L$ is called a polarization if (3.4) holds and $H$, defined by (3.5), is positive definite hermitian. Note that $E = ImH$, so $E$ and $H$ determine each other. The polarization is called principal if in addition, $E$ is equivalent to 
\[ \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \]

A slight refinement of the previous theorem.

**Theorem 3.2.6 (Riemann).** $\mathbb{C}^g/L$ is an abelian variety if $L$ carries a polarization.

We will see later the converse is true.

### 3.3 Jacobians

Let $X$ be a smooth projective curve of genus $g$. Then 
\[ H^1(X, \mathbb{Z}) \cong \mathbb{Z}^{2g} \]

We also proved earlier that $H^1(X, \mathcal{O}_X) \cong \mathbb{C}^g$. The natural map $\mathbb{Z} \to \mathcal{O}_X$ of sheaves induces a map 
\[ H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}_X) \]

**Proposition 3.3.1.** This map is injective and the image is lattice.

**Proof.** Since $H^1(X, \mathbb{Z})$ sits a lattice inside $H^1(X, \mathbb{R})$, it is enough to show that the natural map 
\[ r : H^1(X, \mathbb{R}) \to H^1(X, \mathcal{O}_X) \]
is injective. It $\alpha$ is a harmonic form representing a nonzero element of $H^1(X, \mathbb{R})$. Then we can uniquely decompose $\alpha = \alpha^{0,1} + \alpha^{1,0}$ into a sum of a holomorphic and antiholomorphic forms. Note that $r(\alpha) = \alpha^{0,1}$. Since $\alpha$ is real, $\alpha^{0,1} = \overline{\alpha^{1,0}}$. Therefore $(\alpha) \neq 0$. 
\[ \square \]
We define the Jacobian

\[ J(X) = \frac{H^1(X, \mathcal{O}_X)}{H^1(X, \mathbb{Z})} \]

This is a \( g \) dimensional complex torus. Form the exponential sequence, we see that this fits into a sequence

\[ J(X) \rightarrow H^1(X, \mathcal{O}^*) \rightarrow H^2(X, \mathbb{Z}) \]

In other words, \( J(X) = \text{Pic}^0(X) \) as groups.

Next, we construct a polarization on \( J(X) \). We have a cup product pairing \(-E\) on \( H^1(X, \mathbb{Z}) \) which coincides with the intersection pairing under the Poincaré duality isomorphism

\[ H^1(X, \mathbb{Z}) \cong H^1(X, \mathbb{Z}) \]

In terms of the embedding \( H^1(X, \mathbb{Z}) \subset H^1(X, \mathbb{R}) \), this given by integration

\[ E(\alpha, \beta) = -\int_X \alpha \wedge \beta \]

The key point is that \( E \) is skew symmetric with determinant +1. By linear algebra, we can find a basis for \( L \), called a symplectic basis, so that \( E \) is represented by the matrix

\[ \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \]

We can identify \( H^1(X, \mathcal{O}_X) \subset H^1(X, \mathbb{C}) \) with the subspace spanned by antiholomorphic 1-forms. The are forms given locally by \( \alpha = f(z) \overline{dz} \) where \( f \) is holomorphic. Since

\[ \alpha \wedge \overline{\alpha} = +2|f(z)|^2 dx \wedge dy \]

we conclude that

\[ H(\omega, \eta) = i \int_X \omega \wedge \overline{\eta} \]

is a positive definite Hermitian form on \( H^1(X, \mathcal{O}_X) \). Thus \( E \) defines a principal polarization. Therefore

**Theorem 3.3.2.** The Jacobian \( J(X) \) is a \( g \) dimensional principally polarized abelian variety.

In the first section, we used the dual description of \( J(X) \). Given a complex torus \( T = V/L \), let \( V^\dagger \) denote the space of antilinear maps \( \ell : V \rightarrow \mathbb{C} \) and let

\[ L^\dagger = \{ \ell \in V^\dagger \mid \text{Im} \ell(L) \subseteq L \} \]

The dual torus \( T^* = V^\dagger/L^\dagger \). If \( H \) is a principal polarization, then \( v \mapsto H(v, -) \) gives an isomorphism \( V \cong V^\dagger \) mapping \( L \) isomorphically to \( L^\dagger \). Therefore \( T \cong T^* \). We can apply this to the Jacobian to obtain

\[ J(X) \cong \frac{H^1(X, \mathcal{O})^\dagger}{H^1(X, \mathbb{Z})^\dagger} \cong \frac{H^0(X, \Omega^1_X)^*}{H^1(X, \mathbb{Z})} \]
Using the last description, we define the Abel-Jacobi map as follows. Fix a base point $x_0$. Define

$$AJ : X \to J(X)$$

by sending $x$ to the functional $\int_{x_0}^x$. We extend this to a map

$$AJ : X^n \to J(X)$$

by $AJ(x_1,\ldots,x_n) = \sum AJ(x_i)$. This factors through quotient $S^n X := X/S_n$ by the symmetric group. This can be given the structure of a smooth projective variety. The points of $S^n X$ can be viewed as degree $n$ effective divisors.

**Theorem 3.3.3** (Abel’s theorem). $AJ(D) = AJ(D')$ if and only if $D$ is linearly equivalent to $D'$.

We will prove the reverse direction. We need:

**Lemma 3.3.4.** Any holomorphic map $f : \mathbb{P}^N \to T$ to a complex torus is constant.

*Proof.* Since $\mathbb{P}^N$ is simply connected, $f$ lifts to a holomorphic map from $\mathbb{P}^N$ to the universal cover $\mathbb{C}^n$. Since $\mathbb{P}^n$ is compact, this is constant.

*Proof of theorem.* Fix $D \in S^n X$. The set of effective divisors linearly equivalent to a given degree $n$ divisor $D$ forms a projective space $|D| = \mathbb{P}(H^0(X, \mathcal{O}_X(D)))$. Furthermore, this is a subvariety of $S^n X$. Therefore $AJ(|D|)$ is constant by the lemma.

**Theorem 3.3.5** (Jacobi’s inversion theorem). If $n \geq g$, then $AJ : S^n X \to J(X)$ is surjective.

We won’t prove this, but we will draw some conclusions. Jacobi implies that $AJ : S^g X \to J(X)$ is a surjective map between varieties of the same dimension. Moreover, Abel’s theorem tells us that the fibres of $AJ$ are projective spaces and therefore connected. Therefore $AJ : S^g X \to J(X)$ is birational. Weil gave a purely algebraic construction of $J(X)$ by working backwards from this. First he observed that $S^g X$ has a “birational group law” i.e. it has rational maps which satisfy the identities of a group. Then he showed that any birational group can always be completed to an algebraic group, that is a variety with morphisms that make it into a group. Carrying out the procedure for $S^g X$ yields $J(X)$. This construction works over any algebraically closed field.

## 3.4 More on polarizations

Earlier we showed that polarized tori are abelian varieties. Now we want to show the converse. To do this, we need to understand the geometric meaning of polarizations. Let us start with analyzing the de Rham cohomology of a torus. For the moment, we can ignore the complex structure and work with a real
torus $X = \mathbb{R}^n / \mathbb{Z}^n$. Let $e_1, \ldots, e_n$ (resp. $x_1, \ldots, x_n$) denote the standard basis (resp. coordinates) of $\mathbb{R}^n$. A $k$-form is an expression

$$\alpha = \sum f_{i_1, \ldots, i_k}(x_1, \ldots, x_n)dx_{i_1} \wedge \ldots \wedge dx_{i_k}$$

where the coefficient are $L$-periodic $C^\infty$-functions. We let $\mathcal{E}^k(X)$ denote the space of these. As usual

$$d\alpha = \sum \frac{\partial f_{i_1, \ldots, i_k}}{\partial x_j} dx_j \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k}$$

This satisfies $d^2 = 0$. So we defined $k$-th de Rham cohomology as

$$H^k(X, \mathbb{R}) = \ker[\mathcal{E}^k(X) \xrightarrow{d} \mathcal{E}^{k+1}(X)] / \text{im}[\mathcal{E}^{k-1}(X) \xrightarrow{d} \mathcal{E}^k(X)]$$

The constant form $dx_{i_1} \wedge \ldots \wedge dx_{i_k}$ certainly defines an element of this space, which is nonzero because it has nonzero integral along the subtorus spanned by $e_{i_j}$. (Integration can be interpreted as pairing between cohomology and homology.) These form span by the following special case of Künneth’s formula:

**Theorem 3.4.1.** A basis is given by cohomology classes of constant forms $\{dx_{i_1} \wedge \ldots \wedge dx_{i_k}\}$. Thus $H^k(X, \mathbb{R}) \cong \wedge^k \mathbb{R}^n$.

It is convenient to make this independent of the basis. Let us suppose that we have a torus $X = V/L$ given as quotient of real vector space by a lattice. Then we can identify $\alpha \in H^k(X, \mathbb{R})$ with the alternating $k$-linear map

$$L \times \ldots \times L \to \mathbb{R}$$

sending $(\lambda_1, \ldots, \lambda_k)$ to the integral of $\alpha$ on the torus spanned by $\lambda_j$. Thus we have an natural isomorphism

$$H^k(X, \mathbb{R}) = \wedge^k \text{Hom}(L, \mathbb{R})$$

This works with any choice of coefficients such as $\mathbb{Z}$. For our purposes, we can identify $H^k(X, \mathbb{Z})$ the group of integer linear combinations of constant forms. Then

$$H^k(X, \mathbb{Z}) = \wedge^k L^*, \ L^* = \text{Hom}(L, \mathbb{Z})$$

Now we return to the complex case. Let $V = \mathbb{C}^n$ be a complex vector space, and $X = V/L$ a complex torus. Let $z_1, \ldots, z_n$ be coordinates on $V$. Then $x_1 = \text{Re} z_1, y_1 = \text{Im} z_1, \ldots$ give real coordinates. Suppose that $E : L \times L \to \mathbb{Z}$ is a polarization. By the above, we can view $E \in H^2(X, \mathbb{Z})$. Under the inclusion $H^2(X, \mathbb{Z}) \subset H^2(X, \mathbb{C})$, we can express

$$E = \sum a_{pq} dz_p \wedge dz_q + \sum b_{pq} dz_p \wedge d\bar{z}_q + \sum c_{pq} d\bar{z}_p \wedge d\bar{z}_q$$

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Identifying $V = \mathbb{R} \otimes \mathbb{Z}$ gives a real endomorphism $J$ on the right corresponding to $i$ on the right. Recall that to be a polarization $E$ is required to satisfy

$$E(Ju, Jv) = E(u, v)$$

(3.6)

$$H(u, v) = E(Ju, v) + i E(u, v)$$

is positive definite

(3.7)

Using the fact that $Jdz_p = -i dz_p$ etc., these can be translated into conditions on the differential form. Condition (3.6) says

$$E(\lambda u, \lambda v) = \lambda^2 E(u, v)$$

Therefore

$$E = \sum b_{pq} dz_p \wedge d\bar{z}_q$$

is a $(1, 1)$-form. If we normalize the coefficients as

$$E = \frac{i}{2} \sum h_{pq} dz_p \wedge d\bar{z}_q$$

then condition (3.7) is equivalent to $(h_{pq})$ being a positive definite hermitian matrix. A $(1, 1)$-form locally of this type is called positive.

Now let us now turn to the cohomology of $\mathbb{P}^N$. It is known that $H^2(\mathbb{P}^N, \mathbb{Z}) \cong \mathbb{Z}$. Let us represent the generator by a differential form. Let $z_0, \ldots, z_N$ denote homogeneous coordinates. These represent true coordinates on $\mathbb{C}^{N+1}$. We have a projection $\pi : \mathbb{C}^{N+1} \rightarrow \mathbb{P}^N$. Let

$$\tilde{\omega} = \frac{i}{2\pi} \frac{\partial^2}{\partial z_p \partial \bar{z}_q} \log \left| \left| (z_0, \ldots, z_N) \right| \right|^2$$

We now summarize facts that can be checked by direct calculation. See Griffiths-Harris, pp 31-32 for the first two.

1. $\tilde{\omega}$ is expressible in terms of $z_0/z_i, \ldots, z_N/z_i$ which means that $\tilde{\omega} = \pi^* \omega$ for some 2-form on $\mathbb{P}^N$.

2. $\omega$ is positive.

3. If $\mathbb{P}^1 \subset \mathbb{P}^N$ is the line $z_2 = z_3 = \ldots = 0$. Then $\int_{\mathbb{P}^1} \omega = 1$.

The form $\omega$ is usually called “the Kähler form associated to the Fubini-Study metric”, although we will not explain what these words mean. The key point for us is that $\omega$ is a positive $(1, 1)$-form which represents an element of $H^2(\mathbb{P}^N, \mathbb{Z})$. Now suppose that $X \subset \mathbb{P}^N$ is a complex submanifold. Then $\omega|_X$ is also represents and integral cohomology class. Furthermore $\omega$ is locally $\frac{1}{2} \partial \bar{\partial} f$ for some $\mathcal{C}^\infty$-function $f$ for which the so called Levi form $\frac{\partial^2 f}{\partial z_p \partial \bar{z}_q}$ is positive definite. We see that $\omega|_X$ is locally $\frac{1}{2} \partial \bar{\partial} f|_X$ and the Levi form of $f|_X$ has the same property. Therefore $\omega|_X$ is again positive. In the case where $X$ is a complex torus, this proves:

**Theorem 3.4.2.** An abelian variety must carry a polarization.
Chapter 4

Elliptic Surfaces

4.1 Fibered surfaces

Let us say that a (smooth projective) surface $X$ is fibered if it admits a morphism $f : X \to C$ to (smooth projective) curve, such that all fibres are connected. Using Hartshorne Chap III lemma 10.5, we can see that all but finitely many of the fibres $X_q = f^{-1}(q)$ are smooth curves. Let $\Sigma = \{q_1, \ldots, q_n\} \subset C$ be the set of points for which $X_q$ is singular, and $U = C - \Sigma$. We call $\Sigma$ the discriminant.

A singular fibre $X_q$ is best viewed as a subscheme of $X$, but it’s a bit simpler to view it as a divisor $\sum m_i F_i$ given as the pullback of the divisor $q \in C$.

Given a point $p \in X$, set $q = f(p)$. Then we have a morphism $\mathcal{O}_{C,q} \to \mathcal{O}_{X,p}$, which allows us to view the second ring as a module over the first. Since $\mathcal{O}_{X,p}$ is an integral domain, it is torsion free as an $\mathcal{O}_{C,q}$-module.

Lemma 4.1.1. A torsion free module $M$ over a PID $R$ is flat.

Proof. We can write $M = \varinjlim M_i$, where $M_i$ are finitely generated submodules. Since $M_i$ is necessarily torsion free, it is free by the structure theorem for finitely generated modules over a PID. Therefore $M_i$ is flat. Since $(\varinjlim M_i) \otimes N = \varinjlim (M_i \otimes N)$, and direct limits preserve exactness, $M \otimes -$ is an exact functor. \qed

Corollary 4.1.2. $\mathcal{O}_{X,p}$ is a flat $\mathcal{O}_{C,q}$-module for all points. Therefore $f : X \to Y$ is a flat map.

The significance of this stems from the following fact (see Hartshorne, chap III 9.10).

Theorem 4.1.3. Given a flat map over a connected base, the arithmetic genera of the fibres are constant.

Let us refer to this constant number as the fibre genus. In particular, it follows that the usual genera of the smooth fibres $X_q, q \in U$ are the same. This can be proved by a differential topological argument. Set $X_U = f^{-1}U$.

Proposition 4.1.4. All fibres in $X_U$ are diffeomorphic.
Choose a Riemannian metric on $X_U$. This allows us to split the tangent spaces $T_pX_U$ into a direct sum of the vertical space $T^V_p = \ker df$ and horizontal space $T^H_p = (T^V_p)^\perp$. Call a curve horizontal if all of its tangent vectors lie in the horizontal spaces. Given two points $q, q' \in U$, connect them by a path $\gamma : [0,1] \to U$. For each $p \in X_q$, let $\tilde{\gamma}_p : [0,1] \to X_U$ be unique horizontal lift starting at $p$. Then $\phi(p) = \tilde{\gamma}_p(1)$ gives a diffeomorphism $\phi : X_q \to X_{q'}$.

Refining this idea gives more:

**Theorem 4.1.5** (Ehresmann). $X_U \to U$ is a $C^\infty$ fibre bundle, i.e. there exists a $g$ and an open cover in the usual topology $\{U_i\}$ of $U$, such that $f^{-1}U_i$ is diffeomorphic to $U_i \times F$, where $F$ is compact genus $g$ curve.

The last theorem says that all cohomology groups $H^1(X_q, \mathbb{Z}) = \mathbb{Z}^{2g}$, $q \in U$. However, they can fit together in a nontrivial way. This idea can be measured in precise way using the concept of monodromy. Given an element $\gamma \in \pi_1(U, q)$, we can represent it by a path $\gamma : [0,1] \to U$ with $\gamma(0) = \gamma(1) = q$. We have an associated diffeomorphism $\phi : X_q \cong X_q$ constructed as above. Let $\rho(\gamma) = \phi^* : H^1(X_q, \mathbb{Z}) \cong H^1(X_q, \mathbb{Z})$. This is generally a nontrivial automorphism. This gives a homomorphism

$$\rho : \pi_1(U, q) \to \text{Aut}(H^1(X_q, \mathbb{Z}))$$

called the monodromy representation. This can be constructed directly using sheaf theoretical methods using the direct image $R^1f_*\mathbb{Z}|_U$.

### 4.2 Ruled surfaces

Let us start with the simplest case of a fibered surface, $f : X \to C$ where the fibre genus is zero. We say that $X$ is ruled if $f$ has no singular fibres. Here is a basic construction. Let $V$ be a rank 2 algebraic vector bundle over $C$. Recall that $V$ is given by an open covering $\{U_i\}$ of $C$ in the Zariski topology and a collection of regular functions $g_{ij} \in GL_2(O(U_{ij}))$ satisfying the cocycle identity $g_{ik} = g_{ij}g_{jk}$. This tells us how to glue $U_i \times \mathbb{C}^2$ to $U_j \times \mathbb{C}^2$ to get $V$. Since $GL_2$ acts on the projective line, we can use $g_{ij}$ to glue $U_i \times \mathbb{P}^1$ to $U_j \times \mathbb{P}^1$. This gives a ruled surface that we denote by $P(V) \to C$.

In more geometric terms,

$$P(V) = \{(q, \ell) \mid q \in C, \ell \subset V_q \text{ a 1 dim. subspace}\}$$

**Theorem 4.2.1.** Every ruled surface is given by $P(V)$ for some $V$. Two vector bundles, $V, V'$, give isomorphic ruled surfaces if $V = V' \otimes L$, for some line bundle $L$.

**Proof.** We give a sketch. The first step is to show that a ruled surface $f : X \to C$ is locally trivial in the analytic topology. See Beauville’s Complex Algebraic
Surfaces for a proof. From this, it follows that $X$ is given by gluing $U_i \times \mathbb{P}^1$ to $U_j \times \mathbb{P}^1$ by a cocycle $h_{ij} \in PGL_2(\mathcal{O}(U_{ij})).$ We define $H^1(\{U_i\}, PGL_2(\mathcal{O}_C))$ as the set of cocycles modulo the relation $h_{ij} \sim h_{ij}$ if there exists $\eta_i \in PGL_2(\mathcal{O}(U_i))$ such that $h_{ij}' = \eta_i h_{ij} \eta_j^{-1}.$ This is a set with a distinguished element corresponding to the trivial cocycle $h_{ij} = 1.$ We have an central extension of groups

$$1 \to \mathcal{O}_C^* \to GL_2(\mathcal{O}_C) \to PGL_2(\mathcal{O}_C) \to 1$$

which gives an exact sequence of “pointed sets”

$$\tilde{H}^1(C, GL_2(\mathcal{O}_C)) \to \tilde{H}^1(C, PGL_2(\mathcal{O}_C)) \to H^2(C, \mathcal{O}_C^*)$$

We can see that the group on the right vanishing from the exponential sequence

$$0 = H^2(C, \mathcal{O}_C) \to H^2(C, \mathcal{O}_C^*) \to H^3(C, \mathbb{Z}) = 0$$

Therefore $h_{ij}$ lifts to a cocycle $g_{ij}$ in $GL_2$. One can see directly that $g_{ij}'$ is another lift of the $h_{ij}$, then $g_{ij} = a_{ij} g_{ij}'$ for some some cocycle $a_{ij} \in \mathcal{O}^*(U_{ij}).$

These two statements translate exactly to what the theorem says.

We can use this to classify ruled surfaces over $C = \mathbb{P}^1$.

**Theorem 4.2.2** (Grothendieck). Every vector bundle on $\mathbb{P}^1$ is a sum of line bundles.

**Corollary 4.2.3.** Any ruled surface is given by $F_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1})$ for a unique integer $n \geq 0$.

The surface $F_0 = \mathbb{P}^1 \times \mathbb{P}^1$ and $F_1$ can be shown to be the blow up of $\mathbb{P}^2$ at some (any) point.

### 4.3 Elliptic surfaces: examples

An elliptic surface is a surface with fibre genus 1. For example, if $E$ is an elliptic curve, the product $E \times C \to C$ is an elliptic surface. Although this is somewhat trivial. Let us describe a more interesting class of examples.

**Example 4.3.1.** Let $f, g \in \mathbb{C}[x, y, z]$ be homogeneous cubic polynomial such that $V(f)$ and $V(g)$ are distinct nonsingular cubics in $\mathbb{P}^2$. By Bezout’s theorem $V(f) \cap V(g)$ meet in 9 points counting multiplicity. For simplicity, assume that there are no multiplicities. In other words, that there are really 9 points $p_1, \ldots, p_9$. For $(s, t) \neq 0$, let $E_{[t,s]} = V(tf + sg)$. This gives a family of cubic curves parameterized by $\mathbb{P}^1$. This is called a pencil. For all but finitely many values, $E_t = E_{[t,1]}$ is a nonsingular cubic, and therefore an elliptic curve. It is easy to see that $E_t \neq E_{t'}$ unless $t = t'$. Note that all of these curves contain $p_i$.

We can separate these by considering the surface

$$X = \{(p, t) \in \mathbb{P}^2 \times \mathbb{P}^1 \mid p \in E_t\}$$
Projection
\[ \pi : X \to \mathbb{P}^1 \]
makes this a surface having \( E_t \) as the fibre over \( t \). To get a sense of what this looks like, let us project on the other factor
\[ \psi : X \to \mathbb{P}^2 \]
We have \( \psi^{-1}(p) = \{ t \mid p \in E_t \} \). So \( \psi^{-1}(p_i) = \mathbb{P}^1 \). Suppose that \( p \neq p_i \), for any \( i \). Then we claim \( \phi^{-1}(p) \) consists of exactly one point. If not, then we would have \( E_t \cap E_t' \) consisting of at least 10 points. But this is impossible by Bezout. It follows that \( \psi \) is a birational map to \( \mathbb{P}^2 \). In fact, \( X \) is precisely the blow up of \( \mathbb{P}^2 \) at \( p_1, \ldots, p_9 \).

To proceed further, recall that any elliptic curve can be put in Weierstrass form
\[ y^2 = x^3 + ax + b \]
where \( \Delta = 4a^3 + 27b^2 \neq 0 \). Of course, we describing the affine equation; the projective curve is defined by the corresponding homogeneous equation. Also recall that the isomorphism class of the curve is determined by the \( j \)-invariant
\[ j = \frac{a^3}{\Delta} \]
(where we drop the usual normalization factor for simplicity.) It follows that if we substitute \( a \mapsto \lambda^4 a, b \mapsto \lambda^6 b \), we get an isomorphic curve. The isomorphism is simply given by the change of variable \( x \mapsto \lambda^2 x, y \mapsto \lambda^3 y \).

**Example 4.3.2.** Let
\[
\begin{align*}
  a(t) &= a_n t^n + \ldots + a_0 \\
  b(t) &= b_m t^m + \ldots + b_0
\end{align*}
\]
be polynomials, then
\[ y^2 = x^3 + a(t)x + b(t) \quad (4.1) \]
defines an elliptic surface over the affine \( t \)-line. We can try complete to a surface over \( \mathbb{P}^1 \). Let \( s = t^{-1} \), be the coordinate at \( \infty \), and consider
\[ Y^2 = X^3 + \alpha(s)X + \beta(s) \quad (4.2) \]
where
\[
\begin{align*}
  \alpha(s) &= a_0 s^n + \ldots + a_n = s^n a(t) \\
  \beta(s) &= b_0 s^m + \ldots + b_m = s^m b(t)
\end{align*}
\]
Suppose \( n = 4d, m = 6d \) for some integer \( d \geq 0 \). Then by the above discussion, (4.1) and (4.2) can be patched using \( X = s^{2d}x, Y = s^{3d}y \).

**Example 4.3.3.** The previous construction can be generalized by replacing \( \mathbb{P}^1 \) by any curve \( C \), choosing a line bundle \( L \), such that \( H^0(L^i) \neq 0 \) for \( i = 2, 3 \), and viewing \( x, y, a, b \) as global sections of \( L^2, L^3, L^4, L^6 \) in (4.1)
4.4 Singular fibres

Most elliptic surfaces will have singular fibres. Kodaira, and independently Neron, gave a complete classification of the singular fibres. The problem is local, so we can imagine a family of elliptic curves $E_t$, $0 < |t| < \epsilon$ degenerating to a singular, possible reducible curve $E_0 = \sum n_i C_i$. If we insist that $E_0$ stay a cubic, then the problem is easy:

1. $E_0$ is either an irreducible cubic with a node,
2. an irreducible cubic with a cusp,
3. a union of line and an irreducible conic, which may or may not be tangent,
4. or a union of three lines (there are several subcases).

However, in general, $E_0$ will not remain a cubic, so there are many more cases. In fact, there are infinitely many possibilities for the simple reason that we can always blow up the surface along $E_0$. We can recognize a blow up, because we will see an exceptional curve i.e. $\mathbb{P}^1$ with self intersection $-1$ as one of the components. From now on we will insist the no such curve appears among the fibres. Such an elliptic surface is called (relatively) minimal. In order to visualize the result, we use a dual graph: a vertex corresponds to a curve $C_i$; two vertices if the curves meet.

**Theorem 4.4.1 (Kodaira-Neron).** The set of possible singular fibres of a minimal elliptic surface forms two infinite families:

(M$^I_N$) $M > 0$, $N \geq 0$. $N = 0$ corresponds to a smooth curve, $N = 1$ is a nodal rational curve, and otherwise an $N$-gon of $\mathbb{P}^1$’s. All curves occur with multiplicity $M$.

(I$^N\!_N$) $N \geq 0$. A collection of $N + 5$ $\mathbb{P}^1$’s with dual graph $\tilde{D}_{N+4}$

```
  \bullet
  \circ\circ\circ\circ\circ
  \circ\circ\circ\circ\circ
```

A finite list of exceptional cases described below.

People familiar with Dynkin diagrams, which arise in the study of Lie algebras, will recognize $D_{N+4}$ as the usual Dynkin diagram $D_{N+4}$ plus an extra vertex marked in black. Similarly type (I$^N\!_N$) corresponds to the extend Dynkin diagram $A_N$ given by an $N + 1$ cycle. Removing one of the vertices gives $A_N$. The exceptional cases are the cuspidal rational curve (II), a union of two $\mathbb{P}^1$’s meeting at a point (III), three $\mathbb{P}^1$’s meeting at point (IV) and configurations....
corresponding to the extended Dynkin diagrams \( \tilde{E}_6 \) (IV*), \( \tilde{E}_7 \) (III*), \( \tilde{E}_8 \) (II*) listed below. In brackets, we have indicated Kodaira’s notation.

We will be content to prove the weaker statement that the dual graphs of the singular fibres are the ones given above. The proof can be reduced to the study of quadratic forms. Given a graph \( \Gamma \) (allowing loops and multiple edges) with vertices labelled 1 through \( n \), we associate an \( n \times n \) matrix \( Q = Q_{\Gamma} \) given by

\[
q_{ij} = \begin{cases} 
#\text{edges joining } i \text{ and } j & \text{if } i \neq j \\
-2 + 2(\text{number of loops at } i) & \text{if } i = j
\end{cases}
\]

This represents an integer valued quadratic form. Conversely, we see that any symmetric integer matrix with nonnegative off-diagonal entries and diagonal entries in \( \{-2, 0, 2, \ldots\} \) comes from a graph. The key fact, whose proof can be found in Miranda’s Basic Theory of Elliptic Surfaces pp 13-14, is:

**Proposition 4.4.2.** If \( \Gamma \) is a connected graph such that \( Q_{\Gamma} \) is negative semidefinite with one dimensional kernel, then \( \Gamma \) must be an \( \tilde{A}, \tilde{D}, \tilde{E} \) graph. The converse is true, and \( Q_{\Gamma} \) is negative definite for the ADE graphs.

We want to apply this to elliptic surfaces. First, we define the Néron-Severi group of a surface \( X \) as the image of the divisor class group \( NS(X) = \text{im}[Cl(X) \to H^2(X, \mathbb{Z})] \). The cup product pairing on \( H^2 \) restricts to a pairing on \( NS \) which is compatible with the intersection pairing. The proof of the following basic fact can be found in Griffiths-Harris, or it can be deduced from the Riemann-Roch theorem for surfaces (see Hartshorne, Chap V).

**Theorem 4.4.3** (Hodge index theorem). If \( H \) is ample, then \( H^2 > 0 \). If \( D \in NS(X) \) is a nonzero class such that \( D \cdot H = 0 \), then \( D^2 < 0 \)

**Corollary 4.4.4.** The form restricted to the orthogonal complement \( H^\perp \) is negative definite. More generally, the form on \( D^\perp \) is negative definite for any \( D \) with \( D^2 > 0 \).
Let $X$ be a minimal elliptic surface with a singular fibre $E_0 = \sum n_i C_i$. Let $V \subset NS(X)$ be the span of the $C_i$.

**Theorem 4.4.5.** The restriction of the intersection pairing to $V$ is negative semidefinite with a one dimensional kernel. More precisely, $D \in V$ satisfies $D^2 = 0$, if and only if $D = N E_0$ for some integer $N$.

**Proof.** Note that $E_0$ is equivalent in $H^2(X)$ to $E_t$ for any $t$, because the classes of 0 ant $t$ are equal in $H^2(C)$. By choosing $t \neq 0$, we see that $C_i \cdot E_0 = C_i \cdot E_0 \cdot E_t = 0$. In particular, $E_0^2 = 0$. If $D \in V$ satisfies $D^2 > 0$, then the fact that $D \cdot E_0 = E_0^2 = 0$ contradicts the Hodge index theorem. Therefore $D^2 \leq 0$, which says that the form is negative semidefinite.

Now suppose that $D \in V$ satisfies $D^2 = 0$, but $D$ is not a multiple of $E_0$. Then we can find a rational number $r$, so that $G = D + r E_0$ is a linear combination of $C_i$ all nonzero coefficients, and at least one positive coefficient and at least one negative. Write $G = P - N$, where coefficients of $P, N$ are positive. Since $E_0$ is connected $P \cdot N > 0$. Since $D^2 = 0$ and the previous identities $G^2 = 0$. But

$$G^2 = P^2 - 2P \cdot N + N^2 < P^2 + N^2 \leq 0$$

which is a contradiction. \hfill \Box

We are now almost ready to prove the Kodaira-Neron theorem. The additional thing we need is strong form of the adjunction formula.

**Theorem 4.4.6.** If $C$ is a possibly singular curve on surface $X$, then

$$(K + C) \cdot C = 2p_a(C) - 2$$

where the arithmetic genus $p_a(C)$ is defined by $1 - p_a(C) = \chi(O_C)$.

We prove the following special case of Kodaira-Neron.

**Theorem 4.4.7.** The dual graph of a singular fibre of a minimal elliptic surface is of $\tilde{A}, \tilde{D}, \tilde{E}$ type.

**Proof.** Let $E_0 = \sum n_i C_i$ be a singular fibre, and $E_t$ a smooth fibre. We may suppose that $E_0$ is reducible, otherwise there is nothing to prove. Since $E_t$ is an elliptic curve with $E_t^2 = 0$, the adjunction formula implies

$$K \cdot E_0 = K \cdot E_t = 0$$

Therefore

$$\sum n_i (2p_a(C_i) - 2 - C_i^2) = 0 \quad \text{(4.3)}$$

Suppose that $2p_a(C_i) - 2 - C_i^2 < 0$. Then

$$-2 \leq 2p_a(C_i) - 2 - C_i^2 \leq -1$$

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which forces \( C_i^2 = -1 \) and \( p_a(C_i) = 0 \). The last equation implies \( C_i = \mathbb{P}^1 \), so we get a contradiction to minimality. Therefore all \( 2p_a(C_i) - 2 - C_i^2 \geq 0 \) and so they must be 0 by (4.3). But this only possible if \( C_i \cong \mathbb{P}^1 \) and \( C_i^2 = -2 \). This already shows that the intersection matrix on \( V \) comes from a graph, which is necessarily connected. Therefore theorem 4.4.5 proves the graph is of \( \tilde{A}, \tilde{D}, \tilde{E} \) type.

\[ \square \]

**4.5 The Shioda-Tate formula**

Given a field \( K \), an elliptic curve over \( K \), is a genus one curve defined over \( K \) with distinguished \( K \)-rational point \( 0 \in E(K) \). Here “defined over \( K \)” can be taken to mean the equation is defined over \( K \), and \( E(K) \) is the set of solutions in this field. The condition \( 0 \in E(K) \) allows us to define a group law. This is sometimes called the Mordell-Weil group. If \( K \) is a number field, such as \( \mathbb{Q} \), the we have the following important result that this group is finitely generated. Understanding the rank of this group is an important problem in number theory.

Now let us return to the world of complex surfaces and look for an analogue. We start with an elliptic surface \( f : X \to \mathbb{C} \) with a section \( \sigma_0 \). If we remove the singular fibres and possibly some additional fibres form \( X \), it can be described by a Weierstrass equation

\[
y^2 = x^3 + a(t)x + b(t), \quad a(t), b(t) \in \mathbb{C}(C)
\]

This equation can be viewed as defining an elliptic curve over \( K = \mathbb{C}(C) \) that we call the generic fibre. For people familiar with schemes, this is the same thing as the generic fibre in the sense of scheme theory \( E = X \times_C \text{Spec} \mathbb{C}(C) \). The set \( E(K) \) is the set rational sections, i.e. \( \sigma : C \dashrightarrow X \) such that \( f \circ \sigma = \text{id} \). Since \( C \) is a smooth projective curve any rational section can be completed to a regular section. Since we assumed \( E(K) \neq \emptyset \), it forms a group with \( \sigma_0 \) as the identity. So now we can ask does Mordell-Weil hold? The answer is not always. Given the product \( X = E_0 \times C \), then \( E(K) \) is simply \( E_0 \) which uncountable! However, in a sense this is the only problem. Let us say that \( X \to C \) is non-isotrivial, if the \( j \)-functions of the smooth fibres are not all the same.

**Theorem 4.5.1.** If \( f : X \to C \) is non-isotrivial, then there is an exact sequence

\[
0 \to T \to \text{NS}(X) \to E(K) \to 0
\]

where \( T \) is the span of \( \sigma_0(C) \) and the fibres.

**Corollary 4.5.2.** \( E(K) \) is finitely generated.

**Corollary 4.5.3** (Shioda-Tate formula). If \( r_t \) the number components of the fibre \( X_t \),

\[
\text{rank } E(K) = \text{rank } \text{NS}(X) - 2 - \sum r_t - 1
\]
Before getting to the proof, we need to say a bit more about the $NS(X)$. From the exponential sequence we obtain the exact sequence

$$H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}_X) \to \text{Pic}(X) \to H^2(X, \mathbb{Z})$$

Therefore

$$0 \to \text{Pic}^0(X) \to \text{Pic}(X) \to NS(X) \to 0$$

where $\text{Pic}^0(X)$ is the quotient $H^1(X, \mathcal{O}_X)/H^1(X, \mathbb{Z})$. In fact, this quotient can be shown to be complex torus, and even an abelian variety called the Picard variety. This is a generalization of the Jacobian of a curve. In fact, $\text{Pic}^0(C) = J(C)$. One can see that we have a homomorphism of tori $f^*: J(C) \to \text{Pic}^0(X)$.

The key fact that we is that

**Proposition 4.5.4.** If $X \to C$ is non-isotrivial, $J(C) \to \text{Pic}^0(X)$ is an isomorphism.

This can be easily proved using the Leray spectral sequence, but we won’t go into details.

Finally, we need to say a few words about divisors on $E$. Let $\bar{K}$ be the algebraic closure. The Galois group $G = \text{Gal}(\bar{K}/K)$ acts on the group $E(\bar{K})$, and $E(K) \subset E(\bar{K})$ is the subgroup of elements fixed by this action. A divisor is a finite sum $D = \sum n_i p_i \in \text{Div}(E(\bar{K}))$, $\sigma \in G$ acts on $D$ by sending it to $\sum n_i \sigma(p_i)$. We say that $D$ is defined over $K$ if it is fixed by $G$. Let $\text{Div}(E)(K)$ be the group of these. It is sufficient but not necessary that all $p_i \in E(K)$. If $g$ is a rational function on $E$, then $\text{div}(g) \in \text{Div}(E)(K)$. To be explicit, if $E$ is given by (4.4), $g \in \text{Frac}(\mathbb{C}(C))[x, y]/(y^2 - x^3 - ax - b)$, so $g$ is a rational function on $X$ itself. Since $E(\bar{K})$ is a group, we have a homomorphism $\text{Div}(E(\bar{K})) \to E(\bar{K})$ which restricts to $\text{Div}(E)(K) \to E(K)$. This takes principal divisors to 0 by Abel’s theorem (in a slightly more general form than we stated earlier).

**Proof of theorem.** Given a divisor $D$ on $X$, we can take its restriction $D|_E \in \text{Div}(E)(K)$. Therefore we get a homomorphism $r : \text{Div}(X) \to E(K)$, which factors through $\text{Pic}(X)$. The generic fibre $E$ is also the generic fibre of the $f^{-1}U \to U$ for any nonempty Zariski open set $U \subset C$. It follows that if $D$ is sum of components of the fibres, then $D|_E$ is trivial. In particular, $r$ is trivial on $f^*J(C)$. Therefore $r$ factor through $NS(X) = \text{Pic}(X)/f^*J(C)$. We denoting resulting by $r$ also. Given $\sigma \in E(K)$, it can be viewed as section $\sigma : C \to X$. One can see that $r(\sigma(C)) = \sigma$. Therefore $r$ is surjective.

We saw that anything supported on the fibres lies in $\ker r$. Also the zero section $Z = \sigma_0(C)$ lies in $\ker r$ by definition. So $T \subseteq \ker r$. Suppose that $r(D) = 0$, i.e. suppose $D$ maps to $\sigma_0$. Then, Abel, $D|_E - n\sigma_0 = \text{div}(g)$ for some $n \in \mathbb{Z}$ and rational function $g$ on $E$ (in fact, $n = (D \cdot E_t)$). Since, as observed above, $g$ is a rational function on $X$, which, to avoid confusion, we denote by $G$. Then

$$D - n\sigma_0(D) - \text{div}(G)$$

is zero on $E$, it must be zero on $f^{-1}U$ for some Zariski open $U$. But this says that the class of $D \in T$. 

\[\square\]
Proof of Shioda-Tate. By the theorem

\[ \text{rank } E(K) = \text{rank } NS(X) - \text{rank } T \]

So it suffices to show that

\[ \dim T \otimes \mathbb{R} = 2 + \sum (r_t - 1) \]

We can of course sum over the \( t \)'s for which \( E_t \) has at least two components. For each such \( t \), let \( C_{t,1}, \ldots, C_{t,r_t-1} \) be the set of components of \( E_t \) which don’t meet \( Z \). Let \( F \) denote a smooth fibre. It is easy to see that \( Z, F, C_{t,i} \) spans \( T \). It suffices to check that these are linearly independent. We do this by checking that the intersection matrix

\[ A = \begin{pmatrix} Z^2 & Z \cdot F & \ldots \\ F \cdot Z & F^2 & \ldots \\ \ldots & \ldots & \ldots \end{pmatrix} \]

is nonsingular. With respect to the intersection form, we can decompose \( T \otimes \mathbb{R} \) into an orthogonal direct sum

\[ \langle Z, F \rangle \oplus \bigoplus_t \langle C_{t,1}, C_{t,2}, \ldots \rangle \]

so \( A \) decomposes accordingly into blocks. For the first block, the matrix is

\[ \begin{pmatrix} * & 1 \\ 1 & 0 \end{pmatrix} \]

The remaining blocks come from \( ADE \) graphs, so they are negative definite.

\( \square \)