Chapter 4

Algebraic varieties

4.1 Affine varieties

Let k be a field. Affine n-space $\mathbb{A}^n = \mathbb{A}^n_k = k^n$. It's coordinate ring is simply the ring $R = k[x_1, \ldots, x_n]$. Any polynomial can be evaluated at a point $a \in A^n$ to yield an element $f(a) = ev_a(f) \in k$. This gives a surjective homomorphism $ev_a : R \to k$. Its kernel m_a is a maximal ideal. *Let us suppose from now on that k is algebraically closed.* Then we have converse.

Theorem 4.1.1 (Weak Hilbert Nullstellensatz)**.** *Every maximal ideal of R is of the form m_a.*

Given a subset $S \subseteq R$, let $V(S) = \{a \in \mathbb{A}^n \mid \forall f \in S, f(a) = 0\}$. A set of this form is called algebraic. One can see that $V(S) = V(I) = V(\sqrt{I})$, where I is the ideal generated by S and $\sqrt{I} = \{f \mid \exists n, f^n \in I\}$ is the radical. Thus we as well focus on radical ideals, i.e. ideals such that $\sqrt{I} = I$.

Theorem 4.1.2 (Hilbert's Nullstellensatz)**.** *V sets up a bijection between the set of radical ideals of R and algebraic subsets of $\mathbb{A}^n$. The inverse sends an algebraic set X to $I(X) = \{f \in R \mid \forall a \in X, f(a) = 0\}$.*

Theorem 4.1.3. *$\mathbb{A}^n, \emptyset$ are algebraic. The collection of algebraic subsets of $\mathbb{A}^n$ is closed under arbitrary intersections and finite unions.*

It follows that $\mathbb{A}^n$ carries a topology, called the Zariski topology, such that the $X \subset \mathbb{A}^n$ is closed if and only if it is algebraic. Be aware that unlike the topologies in analysis this topology is very weak. It is not even Hausdorff. A closed subset of $\mathbb{A}^n$ is called irreducible if it is not a union of two proper closed sets.

Exercise 11. *Show that the following are equivalent for a closed subset $X \subset \mathbb{A}^n$.*

1. *X is irreducible.*
2. *Any nonempty open subset of X (with respect to the induced topology) is dense.*

3. $I(X)$ is a prime ideal.

An irreducible subset of $\mathbb{A}^n$ is called an algebraic subvariety. A set is called an affine variety if it is an algebraic subvariety of some affine spaces. We can make the collection of affine varieties into a category as follows. A function $F : \mathbb{A}^n \to \mathbb{A}^m$ is a called regular, if there exists polynomials $f_1, \ldots, f_m \in R$ such that

$$F(a_1, \ldots, a_n) = (f_1(a_1, \ldots, a_n), \ldots, f_m(a_1, \ldots, a_n))$$

Such a map is necessarily continuous with respect to the Zariski topologies. The restriction of a regular function between subvarieties $\mathbb{A}^n \supset X \to Y \subset \mathbb{A}^m$ is also called regular. One can see that the collection of affine varieties and regular maps forms a category. The coordinate ring of an subvariety $X \subset \mathbb{A}^n$ is the quotient $\mathcal{O}(X) = R/I(X)$. Given $f \in \mathcal{O}(X)$ and $a \in X$, define $f(a) = \tilde{f}(a)$, where $\tilde{f} \in R$ is a polynomial representing f. To see that this is well defined, observe that any other representative is of the form $\tilde{f} + g$, where $g \in I(X)$. Therefore $\tilde{f}(a) = (\tilde{f} + g)(a)$. Thus we can view $\mathcal{O}(X)$ as a ring of functions on X. This ring is a finitely generated k-algebra which is an integral domain. We refer such an algebra as an affine domain. The coordinate ring construction gives a contravariant functor between these two categories.

Theorem 4.1.4 (Affine duality)**.** *The functor $\mathcal{O}$ gives an antiequivalence between the category of affine varieties and regular maps and the category of affine domains and k-algebra homomorphisms. In more detail, this means that every affine domain is isomorphic to some $\mathcal{O}(X)$, and*

$$Hom_{AffVar}(X, Y) \cong Hom_{AffDom}(\mathcal{O}(Y), \mathcal{O}(X))$$

Sketch. Given $A \in AffDom$, choose a finite set of say n generators. Then we can write $A \cong k[x_1, \ldots, x_n]/I$ for some necessarily prime ideal I. Therefore $A \cong \mathcal{O}(V(I))$.

Suppose that $X \subset \mathbb{A}^n, Y \subset \mathbb{A}^m$. We have to construct an inverse to the map

$$Hom_{AffVar}(X, Y) \to Hom_{AffDom}(\mathcal{O}(Y), \mathcal{O}(X))$$

Let h be an element of the Hom on the right. This can be lifted to a homomorphism H between polynomial rings as indicated below

$$\begin{tikzcd} k[y_1, \ldots, y_m] \arrow[r, dashed, "H"] \arrow[d] & k[x_1, \ldots, x_n] \arrow[d] \\ \mathcal{O}(Y) \arrow[r, "h"] & \mathcal{O}(X) \end{tikzcd}$$

Let $f_i = H(y_i)$. Then these polynomials define a regular map from $X \to Y$ which gives the inverse. $\square$

4.2 Projective varieties

For various reasons, it is desirable to add points at infinity to $\mathbb{A}^n$. But we want to do this in such a way that algebraic geometry still works. The simplest solution is to pass to projective space

$$\mathbb{P}^n = \mathbb{P}^n_k = \{\text{ lines in } k^{n+1} \text{ passing through } 0\}$$

If we choose coordinate $x_0, \ldots, x_n$ on k^{n+1}, now called homogeneous coordinates, we let $[x_0, \ldots, x_n] \in \mathbb{P}^n$ denote the line spanned by the vector $(x_0, \ldots, x_n) \neq 0$. We embed $\mathbb{A}^n \to \mathbb{P}^n$ by sending $(x_1, \ldots, x_n) \mapsto [x_1, \ldots, 1, \ldots x_n]$ 1 in the ith place. The image is precisely, the set

$$U_i = \{[x_0, \ldots, x_n] \mid x_i \neq 0\}$$

and these sets cover $\mathbb{P}^n$. Usually, we pick the first embedding of $\mathbb{A}^n$, and the remaining points $\mathbb{P}^n - U_0$ are the points at infinity. Given a homogeneous polynomial $f \in k[x_0, \ldots, x_n]$, the sets $V_P(f) = \{[a] \in \mathbb{P}^n \mid f(a) = 0\}$ and $D_P(f) = \mathbb{P}^n - V_P(f)$ can be defined as before, and we again get a topology, called the Zariski topology. An irreducible closed subset is called a projective variety. These are given by homogeneous prime ideals in $k[x_0, \ldots, x_n]$ i.e. by prime ideals generated by homogeneous polynomials. An open subset of a projective variety is called quasiprojective. This includes the previous case of affine varieties.

Exercise 12. *Check that the map $\mathbb{A}^n \cong U_0$ above is a homeomorphism with respect to the Zariski topologies.*

4.3 Abstract Algebraic Varieties

Classical algebraic geometry was concerned exclusively with quasiprojective varieties. But by the mid 20th century, people realized that there were limitations with this approach. Various alternatives were proposed by Chevalley, Weil and Zariski, but one that seemed "right" was introduced by Serre in his landmark 1954 paper FAC="Faiceaux Algebriques Coherents" = "Coherent Algebraic Sheaves". This paved the way for Grothendieck's theory of schemes a few years later (which we won't go into).

Let X be an affine variety $X \subset \mathbb{A}^n$. Then it inherits the Zariski topology. This gives a functor from the category of affine varieties to the category of topological spaces and continuous maps. Given $f \in \mathcal{O}(X)$, let $D(f) = \{a \in X \mid f(a) \neq 0\}$ Clearly this is open, and any open set is a union of such sets.

Lemma 4.3.1. *$D(f)$ is also an affine variety. Under affine duality it corresponds to the localization $\mathcal{O}(X)[1/f]$*

Proof. Embed $D(f) \to \mathbb{A}^{n+1}$ by sending $a \mapsto (a, 1/f(a))$. The image is the algebraic set define by (X) and the new equation $x_{n+1}f(x_1, \ldots, x_n) - 1 = 0$. We get an isomorphism

$$\mathcal{O}(D(f)) \cong \mathcal{O}(X)[x_{n+1}]/(x_{n+1}f - 1) \cong \mathcal{O}(X)[1/f]$$

□

Given $U \subset X$, say that $F : U \to k$ is regular if the restriction of F to any set of the form $D(f)$ is regular. In more explicit terms, this means that F is defined by a rational function in the neighbourhood of any point $a \in U$. Note that it may be necessary to use different rational functions at different points. Since the notion of regularity is local in nature, we can see that if set

$$\mathcal{O}_X(U) = \mathcal{O}(U) = \{\text{regular functions from } U \to k\}$$

Proposition 4.3.2. *$\mathcal{O}_X$ forms a sheaf of rings, called the structure sheaf of X.*

A prevariety over k is a concrete ringed space $(X, \mathcal{O}_X)$ of k-valued functions which is locally isomorphic to an affine variety. More precisely, we have a finite open cover $\{U_i\}$ such that $(U_i, \mathcal{O}_X|_{U_i})$ is isomorphic to an affine variety. The "pre" is because we are missing a separation condition, which plays the role of a Hausdorff condition. Without getting into details, the additional condition we need for X to be a variety is that the diagonal $\Delta = \{(x, x) \mid x \in X\}$ is closed in the suitably defined product $X \times X$. The following lemma, proved in chap 1 §6 of Mumford's Red Book, is sufficient for the examples of interest to us.

Lemma 4.3.3. *A prevariety is a variety if and only if for any two points x_1, x_2, there is an open affine set containing both of them.*

Example 4.3.4. *Let $U \subset X$ be an open subset of an affine variety, then $(U, \mathcal{O}_U)$ is a variety. If $U = \mathbb{A}^2 - \{0\}$, this known to be nonaffine. In outline, one checks that $\mathcal{O}(U) \cong k[x, y]$. If it was affine, then we would have to have $U \cong \mathbb{A}^2$ by duality, but this is not true.*

Example 4.3.5. *Let $X = \mathbb{P}^n$ with its Zariski topology. Given an open $U \subset X$, let $\mathcal{O}_X(U)$ be the algebra generated by rational functions F/G, where f and G are homogeneous of the same degree and $G(a) \neq 0, \forall a \in U$. Using the covering U_i introduced earlier, we check that $(U_i, \mathcal{O}) \cong (\mathbb{A}^n, \mathcal{O})$. We can assume that $i = 0$, and we have already shown that the $U_0 \cong \mathbb{A}^n$ as topological spaces, we just have to check that the functions match. For simplicity, we check only that $\mathcal{O}(\mathbb{A}^n) \cong \mathcal{O}(U_0)$. Given $f \in k[x_1, \ldots, x_n]$ of degree d, $f(x_1/x_0, \ldots, x_n/x_0)$ can be expressed as $F(x_0, \ldots, x_n)/x_0^d$ where F is homogeneous of degree d. This maps $\mathcal{O}(\mathbb{A}^n) \to \mathcal{O}(U_0)$ injectively. To see that it is surjective, let $F/G \in \mathcal{O}(U_0)$. Using the Nullstellensatz, we can see that $G \neq 0$ on U_0 forces it to be of the form cx_0^d.*

The previous example extends to any projective or quasiprojective variety.

4.4 Morphisms

The notion of a regular map between projective varieties is fairly complicated. One might guess that it is a map defined by homogeneous polynomials but in fact there are more complicated. However, using it is easy using the ringed space viewpoint. A map $F : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ from one concrete ringed space to another is a morphism if $F : X \to Y$ is continuous and $F^* f := f \circ F \in \mathcal{O}_X(F^{-1}U)$ whenever $f \in \mathcal{O}_Y(U)$. We have the following fact.

Lemma 4.4.1. *Let $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ be affine varieties then $F : X \to Y$ is regular if and only if it is a morphism in the above sense.*

Proof. Suppose that F is a morphism. Let x_i and y_i denote coordinates on $\mathbb{A}^n$ and $\mathbb{A}^m$ respectively. Lift $F^* y_i \in \mathcal{O}(X)$ to polynomials $f_i \in k[x_1, \ldots, x_n]$. Note that the value of $f_i(a)$, for $a \in X$, is precisely the ith coordinate of $F(a)$. Therefore

$$F(a) = (f_1(a), \ldots, f_m(a))$$

So it is regular.

Conversely, suppose that F is regular and given by the above formula. Then it is clearly continuous. For simplicity we prove that $g \in \mathcal{O}(U) \Rightarrow F^* g \in \mathcal{O}(F^{-1}U)$ when $U = Y$. In this case g is given by a polynomial, and

$$F^* g = g(f_1(x_1, \ldots), \ldots, f_m(x_1, \ldots))$$

□

Exercise 13. *Show that a morphism $F : (\mathbb{R}^n, C^\infty) \to (\mathbb{R}^m, C^\infty)$ is the same thing as a C^∞ map.*

When X and Y are algebraic varieties, we define a morphism or regular map between them to be a morphism of ringed spaces. An element of $\mathcal{O}(X)$ is the same thing as a morphism $X \to \mathbb{A}^1$

Let us consider the simplest nonaffine example.

Proposition 4.4.2. *Let $X = \mathbb{P}^1_k$, then $\mathcal{O}(X) = k$.*

This may seem surprising at first, but it is really an analogue of Liouville's theorem that bounded entire functions are contant. Indeed $\mathbb{P}^1_{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is nothing but the Riemann sphere. A holomorphic function on it is bounded entire.

Proof. Let x_0, x_1 denote the homogeneous coordinates. Let $t = x_1/x_0$ denote the coordinate on the affine line $U_0 = \{x_0 \neq 0\} \cong \mathbb{A}^1$. We have a second affine line $U_1 = \{x_1 \neq 0\}$ with coordinate t^{-1}. A regular function on X restricts to a polynomial in t with is also a polynomial in t^{-1}. This forces it to be constant. □

With a bit more work, we can prove that

Proposition 4.4.3. *$\mathcal{O}(\mathbb{P}^n) = k$ for any n.*

4.5 Stalks

We defined stalks earlier using abstract sheaf theory. Now suppose that X is an algebraic variety, and $x \in X$ is a point, what does the stalk $O_{X,x}$ look like? For this problem, we may as well assume that X is affine with coordinate ring $R = \mathcal{O}(X)$. Then x corresponds to a maximal ideal $m = m_x \subset R$. Since R is an integral domain, we have a field of fractions K. Let S denote the complement of m in R. Let $R_m = R[S^{-1}] \subset K$ denote the subring of elements of the form r/s with $s \in S$. We have a homomorphism $R \to R_m$. We can extend m to an ideal mR_m (which we may denote by m later). Since everything outside this ideal is a unit (invertible), it follows that mR_m is the unique maximal ideal of this ring. Thus R_m is a local ring. The geometric meaning of this is explained by

Lemma 4.5.1. *The stalk $\mathcal{O}_{X,x} \cong R_m$.*

Proof. $s \in S$ means that $s(x) \neq 0$ or that $D(s)$ is a neighbourhood of x. We have an injective map $\mathcal{O}(D(s)) \to \mathcal{O}_{X,x}$. Therefore $r/s \in R_m$ can be mapped injectively to $\mathcal{O}_{X,x}$. Now suppose that $f \in \mathcal{O}_{X,x}$, this is section defined in some neighbourhood of the form $D(s)$. Therefore $f = r/s$, for $r \in R$. So the map is surjective as well.

□

As a corollary, we see that the stalks of $\mathcal{O}_X$ are local rings. We say that $(X, \mathcal{O}_X)$ is a locally ringed space. The other examples of C^∞ and complex manifolds share this property.