

## Mixed Hodge Structures Associated to Geometric Variations

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This is largely an exposition of Morihiko Saito's theory of mixed Hodge modules, which gives a very general framework in which to do Hodge theory. A mixed Hodge module over a point is the same thing as a mixed Hodge structure, and in general the category of these objects possess operations which mirror the standard operations of sheaf theory. Saito's theory has been laid out in a pair of long and densely written papers [S1, S3], and it would be fair to say that the details have not been very widely assimilated. The primary goal of this article is to make these ideas more accessible and explicit.

A variation of Hodge structure gives a good notion of a regular family of Hodge structures. It is natural to extend this to families with singularities. Recall that the constituents of a variation of a Hodge structure are a local system and a compatible filtered vector bundle with connection. As a first step toward the more general notion of a Hodge module, these are replaced by a perverse sheaf and a filtered  $D$ -module respectively. At this stage the challenge for any good theory would be to find a subclass of these pairs which is both robust and meaningful Hodge theoretically. Saito does this by a rather delicate induction. He demands that the "restriction" of such a pair to any subvariety, given by a vanishing cycle construction, should stay in the subclass; and that over a point it determines a Hodge structure in the usual sense. For mixed Hodge modules, Saito also imposes a weight filtration with additional constraints.

The category of mixed Hodge modules possesses direct images. So in particular the cohomology of these modules carry mixed Hodge structures. Therefore one gets a mixed Hodge structure on the cohomology of a variation of Hodge structure. One of my goals here is to describe this as explicitly as possible, and to compare it with the mixed Hodge structure constructed by the author in [A] in geometric examples. The description for the Hodge filtration on it is fairly straightforward, but the weight filtration is somewhat trickier. The key ingredient is a good local description for mixed

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Hodge modules in a neighbourhood of a divisor with normal crossings, which refines the combinatorial model for perverse sheaves on a polydisk [GGM]. With this local model in hand, the weight filtration can be then described in reasonably explicit terms .

Here is a quick synopsis of the contents of the article. The first three sections summarize background material on the Riemann-Hilbert correspondence, vanishing cycles, and standard Hodge theory. All of these ingredients are needed for the theory of pure and mixed Hodge modules discussed in the fourth section. The explicit description of Saito's mixed Hodge structure is given in §4.5. The final section contains the comparison theorem.

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## 1 *D*-modules and Perverse Sheaves

### 1.1 *D*-modules

References for this section are [Bo, HTT, K2]. Suppose that  $X$  is a smooth complex algebraic variety. Given an open set  $U \subseteq X$ , the set of algebraic differential operators  $D_X(U)$  on  $U$  consists of  $\mathbb{C}$ -linear endomorphisms of  $\mathcal{O}_X(U)$  defined inductively as follows. An operator of order 0 is just (multiplication by) a function in  $\mathcal{O}_X(U)$ . A differential operator  $P$  of order at most  $k$  is an endomorphism such that  $[P, f]$  is an operator of order less than  $k$  for any  $f \in \mathcal{O}_X(U)$ . This is a sort of higher Leibnitz rule.  $D_X$  forms a sheaf of noncommutative rings with an increasing filtration  $F_k D_X$  by operators of order at most  $k$ . Every point  $x \in X$  has a Zariski neighbourhood  $U$  such that any element of  $D_X(U)$  can be expanded as a sum

$$\sum f_{j_1 \dots j_n} \partial_1^{j_1} \dots \partial_n^{j_n}$$

where  $\partial_j = \frac{\partial}{\partial x_j}$  with  $x_1, \dots, x_n$  local coordinates, i.e. a minimal set of generators for the maximal ideal at  $x$ . The order then has the usual meaning as the highest value of  $j_1 + \dots + j_n$  occurring with a nonzero coefficient. When  $X = \mathbb{A}^n$ , the ring of global sections of  $D_X$  is the so called Weyl algebra, which is given by generators  $x_1, \dots, x_n, \partial_1, \dots, \partial_n$  and relations

$$\begin{aligned} [x_i, x_j] &= [\partial_i, \partial_j] = 0 \\ [\partial_i, x_j] &= \delta_{ij}. \end{aligned}$$

A standard computation shows that the associated graded for the Weyl algebra

$$Gr(\Gamma(D_{\mathbb{A}^n})) = \bigoplus_k F_k/F_{k-1}$$

is just the polynomial ring in generators  $x_1, \dots, x_n, \partial_1, \dots, \partial_n$ . In general,  $Gr(D_X)$  can be identified with the sheaf of regular functions on the cotangent bundle  $T^*X$  pushed down to  $X$ .

Since  $D_X$  is noncommutative, we have to take care to distinguish between left and right modules over it. We usually mean left. We will primarily be interested in modules which are coherent, i.e. locally finitely presented, over  $D_X$ . A coherent  $D_X$ -module need not be coherent as an  $\mathcal{O}_X$ -module.  $D_X$  itself is such an example. Choosing a coordinate neighbourhood  $U$  as above, we can see that a left  $D_X(U)$ -module is given by an  $\mathcal{O}_X(U)$ -module  $M$  on which  $\partial_1, \dots, \partial_n$  act as mutually commuting  $\mathbb{C}$ -linear operators satisfying the Leibnitz rule  $\partial_j(fm) = \frac{\partial f}{\partial x_j}m + f\partial_j(m)$ . When  $M$  is locally free over  $\mathcal{O}_X$ , this additional structure is equivalent to an integrable connection  $\nabla : M \rightarrow M \otimes \Omega_X^1$  given by  $\nabla m = \sum_j \partial_j m \otimes dx_j$ .

A large class of examples can be given as follows. Let  $X = \mathbb{A}^n$  with coordinates  $x_1 \dots x_n$ ,  $Y = \mathbb{A}^{n+1}$  with an additional coordinate  $x_{n+1}$  and suppose  $i(x_1, \dots, x_n) = (x_1, \dots, x_n, 0)$ . Given a  $D_X$ -module  $M$ , set

$$i_+M = \bigoplus_{j=0}^{\infty} \partial_{n+1}^j M$$

(using Borel's notation [Bo]) where  $\partial_{n+1}^j$  are treated as symbols. This becomes a  $D_Y$ -module via

$$P\partial_{n+1}^j m \mapsto \begin{cases} \partial_{n+1}^j Pm & \text{if } P \in \{x_1, \dots, x_n, \partial_1, \dots, \partial_n\} \\ 0 & \text{if } P = x_{n+1} \\ \partial_{n+1}^{j+1} m & \text{if } P = \partial_{n+1}. \end{cases}$$

The direct image  $i_+M$  of a  $D$ -module along a more general closed immersion can be defined by a similar procedure.

We define a *good* filtration on a  $D_X$ -module  $M$  to be a filtration  $F_p M$  such that

1. The filtration  $F_p M = 0$  for  $p \ll 0$  and  $\cup F_p M = M$ .
2. Each  $F_p M$  is a coherent  $\mathcal{O}_X$ -submodule.
3.  $F_p D_X \cdot F_q M \subseteq F_{p+q} M$ .

**Theorem 1.1** *Let  $M$  be a nonzero coherent  $D_X$ -module. Then*

1.  $M$  possess a good filtration.
2. The support of  $Gr(M)$  in  $T^*X$ , called the characteristic variety, depends only on  $M$ .
3. (Bernstein's inequality) The dimension of the characteristic variety is greater than or equal to  $\dim X$ .

A module  $M$  is called *holonomic* if either it is zero or equality holds in (3), or equivalently if its characteristic variety has dimension less than or equal to  $\dim X$ . The characteristic variety of an integrable connection is the zero section of  $T^*X$ , so it is holonomic.  $D_X$  is not holonomic since its characteristic variety is all of  $T^*X$ . If  $M$  is holonomic, then so is  $i_+M$  for any inclusion. This is easy to see if  $i$  is the inclusion of a point  $p$ , since the characteristic variety of  $i_+M$  is  $T_p$ . A submodule of a holonomic module is again holonomic since its characteristic variety is contained in that of the larger module. Likewise for quotients. More generally:

**Theorem 1.2** *The full subcategory of holonomic modules is closed under submodules, quotients and extensions. Therefore it is abelian. It is also artinian i.e. the descending chain condition holds.*

Recall that an algebraic integrable connection  $\nabla$  on a variety  $X$  has regular singularities if for a suitable compactification, the connection matrix has simple poles along the boundary divisor. One can show that a simple holonomic module always restricts to an integrable connection on a nonempty open subset of its support. This can be used to extend the above notion to holonomic modules. Namely a holonomic module is *regular* if all of its simple subquotients restrict to connections with regular singularities.

Let  $\Omega_{X^{\text{an}}}^p$  denote the sheaf of holomorphic  $p$ -forms on the associated complex manifold  $X^{\text{an}}$ . The sheaf of holomorphic differential operators  $D_{X^{\text{an}}}$  can be defined by the same procedure as for  $D_X$ . We can modify the standard de Rham complex to allow coefficients in any  $D_{X^{\text{an}}}$ -module  $M$ :

$$DR(M)^\bullet = \Omega_{X^{\text{an}}}^\bullet \otimes_{\mathcal{O}_{X^{\text{an}}}} M[\dim X]$$

with differential given by

$$d(dx_{i_1} \wedge \cdots \wedge dx_{i_p} \otimes m) = \sum_j dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p} \otimes \partial_j m.$$

The shift by  $\dim X$  in  $DR$  is done for convenience, as it simplifies various formulas. For example, we have

$$DR(M)^\bullet \cong \mathbb{R}Hom(\mathcal{O}_{X^{\text{an}}}, M)$$

which gives an extension of  $DR$  to the derived category  $D^b(D_X)$ . When  $M$  comes from an integrable connection  $\nabla$ ,  $DR(M)$  gives a resolution of the locally constant sheaf  $\ker(\nabla)$  (up to shift). Conversely any locally constant sheaf arises from such a  $D$ -module. This is the classical version of the Riemann-Hilbert correspondence. For a general  $M$ ,  $DR(M)$  will no longer be a locally constant sheaf in general, but rather a complex with constructible cohomology. Recall that a  $\mathbb{C}_{X^{\text{an}}}$ -module  $L$  is *constructible* if there exists an algebraic stratification of  $X$  such that the restrictions  $L$  to the strata are locally constant with finite dimensional stalks.

**Theorem 1.3 (Kashiwara, Mebkhout)** *The de Rham functor  $DR$  induces an equivalence of categories between the subcategory  $D_{rh}^b(D_X) \subset D^b(D_X)$  of complexes with regular holonomic cohomology and the subcategory  $D_{constr}^b(\mathbb{C}_{X^{\text{an}}}) \subset D^b(\mathbb{C}_{X^{\text{an}}})$  of complexes with constructible cohomology.*

There is more to the story. Various standard sheaf theoretical operations, such as inverse and direct images, correspond to natural operations in the  $D$ -module world. We refer to [Bo, HTT, K2] for further details.

## 1.2 Perverse sheaves

References for this section are [BBD, Br, GoM, HTT]. The category of regular holonomic modules sits in the triangulated category  $D_{rh}^b(D_X)$  as an abelian subcategory. Its image under  $DR$  is the abelian category of complex *perverse sheaves*  $Perv(\mathbb{C}_{X^{\text{an}}})$ . In spite of the name, these objects are neither perverse nor sheaves, but rather a class of well behaved elements of  $D_{constr}^b(\mathbb{C}_{X^{\text{an}}})$ .

**Exercise 1.4**  $DR(\mathcal{O}_X) = \mathbb{C}_X[\dim X]$  is perverse. More generally  $L[\dim X]$  is perverse for any local system.

**Exercise 1.5** Suppose that  $X$  is complete (e.g. projective). Suppose that  $V$  is a vector bundle with an integrable connection  $\nabla : V \rightarrow \Omega_X^1(\log D) \otimes V$  with logarithmic singularities along a normal crossing divisor. Let  $U = X - D$  and  $j : U \rightarrow X$  be the inclusion. Then we have

$$DR(j_*j^*V) = \mathbb{R}j_*(L)[\dim X]$$

is perverse.

**Exercise 1.6** We have the  $i$ th perverse cohomology functor

$${}^p\mathcal{H}^i : D_{constr}^b(X) \rightarrow Perv(\mathbb{C})$$

which corresponds to the functor which assigns the  $i$ th cohomology to a complex of regular holonomic  $D$ -modules.

Perverse sheaves can be characterized by purely sheaf theoretic methods:

**Theorem 1.7**  $F \in D_{\text{constr}}^b(\mathbb{C}_{X^{\text{an}}})$  is perverse if and only if

1. For all  $i$ ,  $\dim \text{supp } \mathcal{H}^{-i}(F) \leq i$ .
2. These inequalities also hold for the Verdier dual

$$D(F) = \mathbb{R}\text{Hom}(F, \mathbb{C}_{X^{\text{an}}}[2 \dim X]).$$

These conditions can be verified directly in example 1.5. The first condition follows from the fact that  $R^i j_* L$  is supported on the union of  $i$ -fold intersections of components of  $D$ . For the second, it is enough to observe that  $D(\mathbb{R}j_* L[\dim X]) = j_! L^\vee[\dim X]$  satisfies (1). Note that the above conditions work perfectly well with other coefficients, such as  $\mathbb{Q}$ , to define full subcategories  $\text{Perv}(\mathbb{Q}_{X^{\text{an}}}) \subset D_{\text{constr}}^b(\mathbb{Q}_{X^{\text{an}}})$ . Perverse sheaves have another source, independent of  $D$ -modules. In the 1970's Goresky and Macpherson introduced intersection homology by a geometric construction by placing restrictions how chains met the singular set in terms of a function referred to as the perversity. Their motivation was to find a theory which behaved like ordinary homology for nonsingular spaces in general; for example, by satisfying Poincaré duality. When their constructions were recast in sheaf theoretic language, they provided basic examples of perverse sheaves.

**Exercise 1.8** Suppose that  $Z \subseteq X$  is a possible singular subvariety. Then the complex  $IC_Z(\mathbb{Q})$  computing the rational intersection cohomology of  $Z$  is (after a suitable shift and extension to  $X$ ) a perverse sheaf on  $X$ . This is more generally true for the complex  $IC_Z(L)$  computing intersection cohomology of  $Z$  with coefficients in a locally constant sheaf defined on a Zariski open  $U \subseteq Z$ . In the notation of [BBD], this would be denoted by  $i_* j_! L[\dim Z]$ , where  $j : U \rightarrow Z$  and  $i : Z \rightarrow X$  are the inclusions. In the simplest case, where  $U \subseteq Z = X$  is the complement of a finite set,  $IC_X(\mathbb{Q}) = \tau_{< \dim X} \mathbb{R}j_* \mathbb{Q}[\dim X]$ , where  $\tau$  is the truncation operator.

The objects  $IC_Z(L)$  have a central place in the theory, since all the simple objects are known to be of this form. It follows that all perverse sheaves can be built up from such sheaves, since the category is artinian. Further details can be found in the references given at the beginning of this section.

## 2 Vanishing Cycles

### 2.1 Vanishing cycles

Vanishing cycle sheaves and their corresponding  $D$ -modules form the basis for Saito's constructions described later. We will start with the classical picture. Suppose that  $f : X \rightarrow \mathbb{C}$  is a morphism from a nonsingular variety. The fiber  $X_0 = f^{-1}(0)$  may be singular, but the nearby fibers  $X_t$ ,  $0 < |t| < \epsilon \ll 1$  are not. The preimage of the  $\epsilon$ -disk  $f^{-1}\Delta_\epsilon$  retracts onto  $X_0$ , and  $f^{-1}(\Delta_\epsilon - \{0\}) \rightarrow \Delta_\epsilon - \{0\}$  is a fiber bundle. Thus we have a monodromy action by the (counterclockwise) generator  $T \in \pi_1(\mathbb{C}^*, t)$  on  $H^i(X_t)$ . (From now on, we will tend to treat algebraic varieties as analytic spaces, and will no longer be scrupulous about making a distinction.) The image of the restriction map

$$H^i(X_0) = H^i(f^{-1}\Delta_\epsilon) \rightarrow H^i(X_t),$$

lies in the kernel of  $T - 1$ . The restriction is dual to the map in homology which is induced by the (nonholomorphic) collapsing map of  $X_t$  onto  $X_0$ ; the cycles which die in the process are the vanishing cycles.

Let us reformulate things in a more abstract way following [SGA7]. The *nearby cycle* functor applied to  $F \in D^b(X)$  is

$$\mathbb{R}\Psi F = i^* \mathbb{R}p_* p^* F,$$

where  $\tilde{\mathbb{C}}^*$  is the universal cover of  $\mathbb{C}^* = \mathbb{C} - \{0\}$ , and  $p : \tilde{\mathbb{C}}^* \times_{\mathbb{C}} X \rightarrow X$ ,  $i : X_0 = f^{-1}(0) \rightarrow X$  are the natural maps. The *vanishing cycle* functor  $\mathbb{R}\Phi F$  is the mapping cone of the adjunction morphism  $i^* F \rightarrow \mathbb{R}\Psi F$ , and hence it fits into a distinguished triangle

$$i^* F \rightarrow \mathbb{R}\Psi F \xrightarrow{\text{can}} \mathbb{R}\Phi F \rightarrow i^* F[1].$$

Both  $\mathbb{R}\Psi F$  and  $\mathbb{R}\Phi F$  are often both loosely referred to as sheaves of vanishing cycles. These objects possess natural monodromy actions by  $T$ . If we give  $i^* F$  the trivial  $T$  action, then the diagram with solid arrows commutes.

$$\begin{array}{ccccccc} i^* F & \longrightarrow & \mathbb{R}\psi_* F & \xrightarrow{\text{can}} & \mathbb{R}\phi_* F & \longrightarrow & i^* F[1] \\ \downarrow & & \downarrow T-1 & & \downarrow \text{var} & & \downarrow \\ 0 & \longrightarrow & \mathbb{R}\psi_* F & \xrightarrow{=} & \mathbb{R}\psi_* F & \longrightarrow & 0. \end{array}$$

Thus we can deduce a morphism  $\text{var}$ , which completes this to a morphism of triangles. In particular,  $T - 1 = \text{var} \circ \text{can}$ . A further diagram chase also shows that  $\text{can} \circ \text{var} = T - 1$ .

Given  $p \in X_0$ , let  $B_\epsilon$  be an  $\epsilon$ -ball in  $X$  centered at  $p$ . Then  $f^{-1}(t) \cap B_\epsilon$  is the so called Milnor fiber. The stalks

$$\mathcal{H}^i(\mathbb{R}\Psi\mathbb{Q})_p = H^i(f^{-1}(t) \cap B_\epsilon, \mathbb{Q})$$

$$\mathcal{H}^i(\mathbb{R}\Phi\mathbb{Q})_p = \tilde{H}^i(f^{-1}(t) \cap B_\epsilon, \mathbb{Q})$$

give the (reduced) cohomology of the Milnor fiber. And

$$H^i(X_0, \mathbb{R}\Psi\mathbb{Q}) = H^i(f^{-1}(t), \mathbb{Q})$$

is, as the terminology suggests, the cohomology of the nearby fiber. We have a long exact sequence

$$\dots H^i(X_0, \mathbb{Q}) \rightarrow H^i(X_t, \mathbb{Q}) \xrightarrow{can} H^i(X_0, \mathbb{R}\Phi\mathbb{Q}) \rightarrow \dots$$

The following is a key ingredient in the whole story [BBD, Br]:

**Theorem 2.1 (Gabber)** *If  $L$  is perverse, then so are  $\mathbb{R}\Psi L[-1]$  and  $\mathbb{R}\Phi L[-1]$ .*

We set  ${}^p\psi_f L = {}^p\psi L = \mathbb{R}\Psi L[-1]$  and  ${}^p\phi_f L = {}^p\phi L = \mathbb{R}\Phi L[-1]$ .

## 2.2 Perverse Sheaves on a polydisk

Let  $\Delta$  be a disk with the standard coordinate function  $t$ , and inclusion  $j : \Delta - \{0\} = \Delta^* \rightarrow \Delta$ . Fix  $t_0 \in \Delta^*$ . Consider a perverse sheaf  $F$  on  $\Delta$  which is locally constant on  $\Delta^*$ . Then we can form the diagram

$${}^p\psi_{t_0} F \xrightleftharpoons[var]{can} {}^p\phi_{t_0} F$$

Note that the objects in the diagram are perverse sheaves on  $\{0\}$  i.e. vector spaces. The space on the left is nothing but the stalk of  $F$  at  $t_0$ . The space on the right can also be made concrete in various cases given below. Observe that since  $T = I + var \circ can$  is the monodromy of  $F|_{\Delta^*}$ , it is invertible. This leads to the following elementary description of the category due to Deligne and Verdier (c.f. [V, §4]).

**Proposition 2.2** *The category of perverse sheaves on the disk  $\Delta$  which are locally constant on  $\Delta^*$  is equivalent to the category of quivers of the form*

$$\psi \xrightleftharpoons[v]{c} \phi$$

(i.e. finite dimensional vector spaces  $\phi, \psi$  with maps as indicated) such that  $I + v \circ c$  is invertible.

It is instructive to consider some basic examples. We see immediately that

$$0 \begin{array}{c} \longleftarrow \\ \longrightarrow \end{array} V$$

corresponds to the sky scraper sheaf  $V_0$ . Now suppose that  $T$  is an automorphism of a vector space  $V$ . This determines a local system  $L$  on  $\Delta^*$ . Then the perverse sheaf  $j_*L[1]$  corresponds to

$$V \begin{array}{c} \xrightarrow{T-I} \\ \xleftarrow{v} \end{array} \text{im}(T-I)$$

where  $v$  is the inclusion. The perverse sheaf  $\mathbb{R}j_*L[1]$  corresponds to

$$V \begin{array}{c} \xrightarrow{T-I} \\ \xleftarrow{id} \end{array} V$$

The above description can be extended to polydisks  $\Delta^n$  [GGM]. For simplicity, we spell this out only for  $n = 2$ . Let  $t_i$  denote the coordinates. Then we can attach to any perverse sheaf  $F$ , four vector spaces  $V_{11} = {}^p\psi_{t_1} {}^p\psi_{t_2} F$ ,  $V_{12} = {}^p\phi_{t_1} {}^p\psi_{t_2} F \dots$  along with maps induced by  $can$  and  $var$ . The set of these maps have additional commutivity and invertibility constraints, such as  $can_{t_1} \circ can_{t_2} = can_{t_2} \circ can_{t_1}$  etcetera.

**Theorem 2.3** *The category of perverse sheaves on the polydisk  $\Delta^2$  which are constructible for the stratification  $\Delta^2 \supset \Delta \times \{0\} \cup \{0\} \times \Delta \supset \{(0,0)\}$  is equivalent to the category of commutative quivers of the form*

$$\begin{array}{ccc} V_{11} & \begin{array}{c} \xrightarrow{c} \\ \xleftarrow{v} \end{array} & V_{12} \\ \begin{array}{c} \uparrow \\ \downarrow \end{array} \begin{array}{c} c \\ c \end{array} & & \begin{array}{c} \uparrow \\ \downarrow \end{array} \begin{array}{c} c \\ c \end{array} \\ V_{21} & \begin{array}{c} \xrightarrow{c} \\ \xleftarrow{v} \end{array} & V_{22} \end{array}$$

for which  $I + v \circ c$  is invertible.

It will be useful to characterize the subset of intersection cohomology complexes among all the perverse ones. On  $\Delta$  these are either sky scraper sheaves  $V_0$  in which case  $\phi = \ker(v)$ , or sheaves of the form  $j_*L[1]$  for which  $\phi = \text{im}(c)$ . On  $\Delta^2$  (and more generally), we have:

**Lemma 2.4** *A quiver corresponds to a direct sum of intersection cohomology complexes if and only if*

$$V_{ij} = \text{im}(c) \oplus \ker(v)$$

holds for every subdiagram

$$V_{i'j'} \begin{array}{c} \xrightarrow{c} \\ \xleftarrow{v} \end{array} V_{ij}$$

### 2.3 Kashiwara-Malgrange filtration

In this section, we give the  $D$ -module analogue of vanishing cycles (c.f. [PS, §14.2], [S1, §3]). But first we start with a motivating example.

**Exercise 2.5** Let  $Y = \mathbb{C}$  with coordinate  $t$ . Let  $M = \bigcup_m \mathcal{O}_{\mathbb{C}}^n[t^{-m}]$  with the standard basis  $e_i$ . We make this into a  $D_{\mathbb{C}}$ -module by letting  $\partial_t$  act on the left through the singular connection  $\nabla = \frac{d}{dt} + A \frac{dt}{t}$ , where  $A$  is a complex matrix with rational eigenvalues. There is no loss of generality in assuming that  $A$  is in Jordan canonical form. For each  $\alpha \in \mathbb{Q}$ , define  $V_{\alpha}M \subseteq M$  to be the  $\mathbb{C}$ -span of  $\{t^m e_i \mid m \in \mathbb{Z}, m + a_{ii} \geq -\alpha\}$  where  $e_i$  is the standard basis of  $M$ . The following properties are easy to check:

1. The filtration is exhaustive and left continuous:  $\cup V_{\alpha}M = M$  and  $V_{\alpha+\epsilon}M = V_{\alpha}M$  for  $0 < \epsilon \ll 1$ .
2.  $\partial_t V_{\alpha}M \subseteq V_{\alpha+1}M$ , and  $tV_{\alpha}M \subseteq V_{\alpha-1}M$ .
3. The associated graded

$$Gr_{\alpha}^V M = V_{\alpha}M / V_{\alpha-\epsilon}M = \bigoplus_{\{i \mid \alpha + a_{ii} \in \mathbb{Z}\}} \mathbb{C} t^{-\alpha - a_{ii}} e_i$$

is isomorphic to the  $-\alpha$  generalized eigenspace of  $A$ . Moreover, the action of  $t\partial_t$  on this space is identical to the action of  $A$ .

(3) implies that the set of indices where  $V_{\alpha}M$  jumps is discrete. (Such a filtration is called discrete.)

Let  $f : X \rightarrow \mathbb{C}$  be a holomorphic function, and let  $i : X \rightarrow X \times \mathbb{C} = Y$  be the inclusion of the graph. Let  $t$  be the coordinate on  $\mathbb{C}$ , and let

$$V_0 D_Y = D_{X \times \{0\}}\text{-module generated by } \{t^i \partial_t^j \mid i \geq j\}$$

This is the subring of differential operators preserving the ideal  $(t)$ . Let  $M$  be a regular holonomic  $D_X$ -module. It is called *quasiunipotent* along  $X_0 = f^{-1}(0)$  if  ${}^p\psi_f(DR(M))$  is quasiunipotent with respect to the action of  $T \in \pi_1(\mathbb{C}^*)$ . Set  $\tilde{M} = i_+ M$ .

**Theorem 2.6 (Kashiwara, Malgrange)** *There exists at most one filtration  $V_{\bullet} \tilde{M}$  on  $\tilde{M}$  indexed by  $\mathbb{Q}$ , such that*

1. The filtration is exhaustive, discrete and left continuous.
2. Each  $V_\alpha \tilde{M}$  is a coherent  $V_0 D_Y$ -submodule.
3.  $\partial_t V_\alpha \tilde{M} \subseteq V_{\alpha+1} \tilde{M}$ , and  $tV_\alpha \tilde{M} \subseteq V_{\alpha-1} \tilde{M}$  with equality for  $\alpha < 0$ .
4.  $Gr_\alpha^V \tilde{M}$  is a generalized eigenspace of  $t\partial_t$  with eigenvalue  $-\alpha$ .

If  $M$  is quasiunipotent along  $X_0$ , then  $V_\bullet \tilde{M}$  exists.

The module  $\tilde{M}$  rather than  $M$  is used to avoid problems caused by the singularities of  $f$ . When  $f$  is smooth, as in Example 2.5, we can and usually will construct the  $V$ -filtration directly on  $M$ .

Given a perverse sheaf  $L$  and  $\lambda \in \mathbb{C}$ , let  ${}^p\psi_{f,\lambda}L$  and  ${}^p\phi_{f,\lambda}L$  be the generalized  $\lambda$ -eigensheaves of  ${}^p\psi_f L$  and  ${}^p\phi_f L$  under the  $T$ -action. The maps *can* and *var* respect this decomposition into eigensheaves. It is technically convenient at this point to define a modification *Var* of *var* such that  $\text{can} \circ \text{Var}$  and  $\text{Var} \circ \text{can}$  equals  $N = \frac{1}{2\pi\sqrt{-1}} \log T_u$ , where  $T_u$  is the unipotent part of  $T$  in the multiplicative Jordan decomposition. After observing that on each eigensheaf

$$\log T_u = \frac{1}{\lambda} N' - \frac{1}{2} \left( \frac{1}{\lambda} N' \right)^2 + \dots$$

where  $N' = T - \lambda$ , a formula for *Var* is easily found by setting

$$\text{Var} = a_0 \text{var} + a_1 \text{var} \circ N' + a_2 \text{var} \circ (N')^2 + \dots,$$

and solving for the coefficients.

**Exercise 2.7** Continuing with Example 2.5. Let  $L = DR(M)$ . Then  ${}^p\psi_t L = \ker e^* \nabla$ , where  $t = e(\tau) = \exp(2\pi\sqrt{-1}\tau)$ . A basis for this is given by the columns of the fundamental solution of  $e^* \nabla = 0$  which is  $z = \exp(-2\pi\sqrt{-1}A\tau)$ . This has monodromy given by  $T = \exp(-2\pi\sqrt{-1}A)$ . Then  $Gr_\alpha^V M$  is the  $-\alpha$  generalized eigenspace of  $A$ , and this is isomorphic to the  $\lambda$  generalized eigenspace of  $T$  where  $\lambda = e(\alpha)$ .

**Exercise 2.8** In the previous example, when  $n = 1$ ,  $A = 0$ , the perverse sheaf corresponding to  $M$  is  $L = \mathbb{R}j_* \mathbb{C}_{\mathbb{C}^*}$ . We can see that  $Gr_1^V \tilde{M} = \mathbb{C}t^{-1} \cong {}^p\phi_t L$ . For the submodule  $\mathcal{O}_{\mathbb{C}} \subset M$ ,  $DR(\mathcal{O}) = \mathbb{C}$  and  $Gr_1^V \mathcal{O} = 0 \cong {}^p\phi_t \mathbb{C}$ .

**Theorem 2.9 (Kashiwara, Malgrange)** *Suppose that  $L = DR(M)$ . Let  $\alpha \in \mathbb{Q}$  and  $\lambda = e(\alpha)$ . Then*

$$DR(Gr_\alpha^V \tilde{M}) = \begin{cases} {}^p\psi_{f,\lambda}L & \text{if } \alpha \in [0, 1) \\ {}^p\phi_{f,\lambda}L & \text{if } \alpha \in (0, 1] \end{cases}$$

The endomorphisms  $t\partial_t + \alpha, \partial_t, t$  on the left corresponds to  $N, \text{can}, \text{Var}$  respectively, on the right.

A proof is given in [S1, §3.4]. Note that some care is needed in comparing formulas here and in [loc. cit.]. There are shifts in indices according to whether one is working with left or right  $D$ -modules, and decreasing or increasing  $V$ -filtrations.

### 3 Pre-Saito Hodge Theory

#### 3.1 Hodge structures

A (rational) *pure Hodge structure* of weight  $m \in \mathbb{Z}$  consists of a finite dimensional vector space  $H_{\mathbb{Q}}$  with a bigrading

$$H = H_{\mathbb{Q}} \otimes \mathbb{C} = \bigoplus_{p+q=m} H^{pq}$$

satisfying  $\bar{H}^{pq} = H^{qp}$ . Such structures arise naturally from the cohomology of compact Kähler manifolds. For smooth projective varieties, we have further constraints namely the existence of a polarization on its cohomology. A polarization on a weight  $m$  Hodge structure  $H$  is quadratic form  $Q$  on  $H_{\mathbb{Q}}$  satisfying the Hodge-Riemann relations:

$$Q(u, v) = (-1)^m Q(v, u) \quad (3.1)$$

$$Q(H^{pq}, H^{p'q'}) = 0, \text{ unless } p = q', q = p' \quad (3.2)$$

$$\sqrt{-1}^{p-q} Q(u, \bar{u}) > 0, \text{ for } u \in H^{pq}, u \neq 0 \quad (3.3)$$

Given a Hodge structure of weight  $m$ , its Hodge filtration is the decreasing filtration

$$F^p H = \bigoplus_{p' \geq p} H^{p', m-p'}$$

The decomposition can be recovered from the filtration by

$$H^{pq} = F^p \cap \bar{F}^q$$

Deligne extended Hodge theory to singular varieties. The key definition is that of a *mixed Hodge structure*. This consists of a bifiltered vector space  $(H, F, W)$ , with  $(H, W)$  defined over  $\mathbb{Q}$ , such that  $F$  induces a pure Hodge structure of weight  $k$  on  $Gr_k^W H$ . Note that by tradition  $F^\bullet$  is a decreasing filtration denoted with superscripts, while  $W_\bullet$  is increasing. Whenever necessary, we can interchange increasing and decreasing filtrations by  $F_\bullet = F^{-\bullet}$ . A pure Hodge structure of weight  $k$  can be regarded as a mixed Hodge structure such that  $Gr_k^W H = H$ . A mixed Hodge structure is *polarizable* if each  $Gr_k^W H$  admits a polarization. Mixed Hodge structures

form a category in the obvious way. Morphisms are rational linear maps preserving filtrations.

**Theorem 3.1 (Deligne [De3])** *The category of mixed Hodge structures is a  $\mathbb{Q}$ -linear abelian category with tensor product. The functors  $Gr_k^W$  and  $Gr_F^p$  are exact.*

The main examples are provided by:

**Theorem 3.2 (Deligne [De3])** *The singular rational cohomology of a complex algebraic variety carries a canonical polarizable mixed Hodge structure.*

Finally, recall that there is a unique one dimensional pure Hodge structure  $\mathbb{Q}(j)$  of weight  $-2j$  up to isomorphism. By convention, the lattice is taken to be  $(2\pi\sqrt{-1})^j\mathbb{Q}$ . The  $j$ th Tate twist  $H(j) = H \otimes \mathbb{Q}(j)$ .

### 3.2 Variations of Hodge structures

Griffiths introduced the notion a *variation of Hodge structure* to describe the cohomology of family of varieties  $y \mapsto H^m(X_y)$ , where  $f : X \rightarrow Y$  is a smooth projective map. A variation of Hodge structure of weight  $m$  on a complex manifold  $Y$  consists of the following data [Sc]:

- A locally constant sheaf  $L$  of  $\mathbb{Q}$  vector spaces with finite dimensional stalks.
- A vector bundle with an integrable connection  $(E, \nabla)$  plus an isomorphism  $DR(E) \cong L \otimes \mathbb{C}[\dim Y]$ .
- A filtration  $F^\bullet$  of  $E$  by subbundles satisfying Griffiths' transversality:  $\nabla(F^p) \subseteq F^{p-1}$ .
- The data induces a pure Hodge structure of weight  $m$  on each of the stalks  $L_y$ .

A polarization on a variation of Hodge structure is a flat pairing  $Q : L \times L \rightarrow \mathbb{Q}$  inducing polarizations on the stalks. The main example is:

**Exercise 3.3** If  $f : X \rightarrow Y$  is smooth and projective,  $L = R^m f_* \mathbb{Q}$  underlies a polarizable variation of Hodge structure of weight  $m$ .  $E = \mathbb{R}^m f_* \Omega_{X/Y}^\bullet \cong \mathcal{O}_Y \otimes L$  is the associated vector bundle with its Gauss-Manin connection and Hodge filtration  $F^p = \mathbb{R}^m f_* \Omega_{X/Y}^{\geq p}$ .

The next class of examples is of fundamental importance.

**Exercise 3.4** An  $n$  dimensional nilpotent orbit of weight  $m$  consists of

- a filtered complex vector space  $(H, F)$  where  $H$  carries a real structure,
- a real form  $Q$  on  $H$  satisfying the first two Hodge-Riemman relations,
- $n$  mutually commuting nilpotent infinitesimal isometries  $N_i$  of  $(H, Q)$  such that  $N_i F^p \subseteq F^{p-1}$ ,
- there exist  $c > 0$  such that for  $\text{Im}(z_i) > c$ ,  $(H, \exp(\sum z_i N_i)F, Q)$  is a polarized Hodge structure of weight  $m$

Then the filtered polarized bundle with fibre  $(H, \exp(\sum z_i N_i)F, Q)$  descends to a real holomorphic variation of Hodge structure on a punctured polydisk  $(\Delta - \{0\})^n$ . The monodromy around the  $i$ th axis is precisely  $\exp(N_i)$ .

Schmid [Sc, §4] showed that any polarized variation of Hodge structure on a punctured polydisk can be very well approximated by a nilpotent orbit with the same asymptotic behaviour. Thus the local study of a variation of Hodge structure can be reduced to nilpotent orbits. When the local monodromies are unipotent (which can always be achieved by passing to a branched covering), an approximating nilpotent orbit  $(H, F, N_i)$  can be constructed in canonical fashion:  $H$  is the fibre of the Deligne's canonical extension [De2] at 0. The Hodge bundles extend to subbundles of the canonical extension and  $F$  is their fibre at 0.  $N_i$  are the logarithms of local monodromy.

The key analytic fact which makes the rest of the story possible is the following theorem (originally proved by Zucker [Z] for curves).

**Theorem 3.5 [Cattani-Kaplan-Schmid [CKS], Kashiwara-Kawai [KK]]** *Let  $X$  be the complement of a divisor with normal crossings in a compact Kähler manifold. Then intersection cohomology with coefficients in a polarized variation of Hodge structure on  $X$  is isomorphic to  $L^2$  cohomology for a suitable complete Kähler metric on the  $X$ .*

It should be noted that in contrast to the compact case,  $L^2$  cohomology of a noncompact manifold is highly sensitive to the choice the metric, and that it could be infinite dimensional. Here the metric is chosen to be asymptotic to a Poincaré metric along transverse slices to the boundary divisor. From the theorem, it follows that  $L^2$  cohomology is finite dimensional in this case. When combined with the Kähler identities, we get

**Corollary 3.6** *Intersection cohomology  $IH^i(X, L) = H^{i-n}(X, j_* L[n])$  with coefficients in a polarized variation of Hodge structure of weight  $k$  carries a natural pure Hodge structure of weight  $i + k$ .*

### 3.3 Monodromy filtration

We start with a result in linear algebra which is crucial for later developments.

**Proposition 3.7 (Jacobson, Morosov)** *Fix an integer  $m$ . Let  $N$  be a nilpotent endomorphism of a finite dimensional vector space  $E$  over a field of characteristic 0. Then there is a unique filtration*

$$0 \subseteq M_{m-1} \subset \cdots \subset M_m \subseteq \cdots \subseteq M_{m+l} = E$$

called the monodromy filtration on  $E$  centered at  $m$  (in symbols  $M = M(N)[m]$ ), characterized by following properties:

1.  $N(M_k) \subseteq M_{k-2}$
2.  $N^k$  induces an isomorphism  $Gr_{m+k}^M(E) \cong Gr_{m-k}^M(E)$

In applications  $N$  comes from monodromy, which explains the terminology. The attribution is a bit misleading since it is really a corollary of the Jacobson- Morosov theorem. However, it can be given an elementary linear algebra proof (c.f. [Sc, 6.4]), which makes it clear that it applies to any artinian  $\mathbb{Q}$  or  $\mathbb{C}$ -linear abelian category such as the categories of perverse sheaves or regular holonomic modules. The last part of the proposition is reminiscent of the hard Lefschetz theorem. There is an analogous decomposition into primitive parts:

**Corollary 3.8**

$$Gr_k^M(E) = \bigoplus_i N^i PGr_{k+2i}^M(E)$$

where

$$PGr_{m+k}^M(E) = \ker[N^{k+1} : Gr_{m+k}^M(E) \rightarrow Gr_{m-k-2}^M(E)]$$

Returning to Hodge theory,  $M$  gives the weight filtration of Schmid's limit mixed Hodge structure [Sc], which is a natural mixed Hodge structure on the space of nearby cycles. We combine this with the converse statement of Cattani and Kaplan [CK].

**Theorem 3.9 (Schmid, Cattani-Kaplan)** *Suppose that  $(H, F, N_i, Q)$  is a collection satisfying all but the last condition of Example 3.4. Then this is a nilpotent orbit of weight  $m$  if and only all of the following hold:*

1.  $M = M(\sum N_i)[m] = M(\sum t_i N_i)[m]$  for any  $t_i > 0$ .
2.  $(H, M)$  is a real mixed Hodge structure.

3. This structure is polarizable; more specifically, the form  $Q(\cdot, N^k \cdot)$  gives a polarization on the primitive Hodge structures  $PGr_{m+k}^M(H)$ .

Given a filtered vector space  $(V, W)$  with nilpotent endomorphism  $N$  preserving  $W$ , a *relative monodromy* filtration is a filtration  $M$  on  $V$  such that

- $NM_k \subseteq M_{k-2}$
- $M$  induces the monodromy filtration centered at  $k$  on  $Gr_k^W V$  with respect to  $N$ .

It is known that  $M$  is unique if it exists (c.f. [SZ]), although it need not exist in general. In geometric situations, the existence of relative weight filtrations was first established by Deligne using  $\ell$ -adic methods.

### 3.4 Variations of mixed Hodge structures

A *variation of mixed Hodge structure* consists of:

- A locally constant sheaf  $L$  of  $\mathbb{Q}$  vector spaces with finite dimensional stalks.
- An ascending filtration  $W \subseteq L$  by locally constant subsheaves
- A vector bundle with an integrable connection  $(E, \nabla)$  plus an isomorphism  $DR(E) \cong L \otimes \mathbb{C}[\dim Y]$ .
- A filtration  $F^\bullet$  of  $E$  by subbundles satisfying Griffiths' transversality:  $\nabla(F^p) \subseteq F^{p-1}$ .
- $(Gr_m^W(L), \mathcal{O}_X \otimes Gr_m^W(L), F^\bullet(\mathcal{O}_X \otimes Gr_m^W(L)))$  is a variation of pure Hodge structure of weight  $m$ .

It follows that this data induces a mixed Hodge structure on each of the stalks  $L_y$ . Steenbrink and Zucker [SZ] showed that additional conditions are required to get a good theory. While these conditions are rather technical, they do hold in most natural examples.

A variation of mixed Hodge structure over a punctured disk  $\Delta^*$  is *admissible* if

- The pure variations  $Gr_m^W(L)$  are polarizable.
- There exists a limit Hodge filtration  $\lim_{t \rightarrow 0} F_t^p$  compatible with the one on  $Gr_m^W(L)$  constructed by Schmid.
- There exist a relative monodromy filtration  $M$  on  $(E = L_t, W)$  with respect to the logarithm  $N$  of the unipotent part of monodromy.

For a general base  $X$  the above conditions are required to hold for every restriction to a punctured disk [K1]. Note that pure polarizable variations of Hodge structure are automatically admissible by [Sc]. A local model for admissible variations is provided by a *mixed nilpotent orbit* called an infinitesimal mixed Hodge module in [K1]. This consists of:

- A bifiltered complex vector space  $(H, F, W)$  with  $(H, W)$  defined over  $\mathbb{R}$ ,
- commuting real nilpotent endomorphisms  $N_i$  satisfying  $N_i F^p \subseteq F^{p-1}$  and  $N_i W_k \subseteq W_k$ ,

such that

- $Gr_k^W$  is a nilpotent orbit of weight  $k$  for some choice of form.
- The monodromy  $M$  filtration exists for any partial sum  $\sum N_{i_j}$ , and  $N_{i_j} M_k \subseteq M_{k-2}$  holds for each  $i_j$ .

With an obvious notion of morphism, mixed nilpotent orbits of a given dimension forms a category which turns out to be abelian [K1, 5.2.6].

## 4 Hodge Modules

### 4.1 Hodge modules on a curve

We are now ready to begin describing Saito's category of Hodge modules in a special case. We start with the category  $MF_{rh}(D_X, \mathbb{Q})$  whose objects consist of

- A perverse sheaf  $L$  over  $\mathbb{Q}$ .
- A regular holonomic  $D_X$ -module  $M$  with an isomorphism  $DR(M) \cong L \otimes \mathbb{C}$ .
- A good filtration  $F$  on  $M$ .

Morphisms are compatible pairs of morphisms of  $(M, F)$  and  $L$ . Variations of Hodge structure give examples of such objects (note that Griffiths' transversality  $\partial_{x_i} F_p \subseteq F_{p+1}$  is needed to verify (3)). The category  $MF_{rh}(X, \mathbb{Q})$  is really too big to do Hodge theory, and Saito defines the subcategory of (polarizable) Hodge modules which provides a good setting. The definition of this subcategory is extremely delicate, so as a warm up we will give a direct description for Hodge modules on a smooth projective curve  $X$  (fixed for the remainder of this section).

Given an inclusion of a point  $i : \{x\} \rightarrow X$  and a polarizable pure Hodge structure  $(H, F, H_{\mathbb{Q}})$  of weight  $k$ , the  $D$ -module pushforward  $i_+ H$  with the

filtration induced by  $F$ , together with the skyscraper sheaf  $H_{\mathbb{Q},x}$  defines an object of  $MF_{rh}(X)$ . Let us call these polarizable Hodge modules of type 0 and weight  $k$ .

Given a polarizable variation of Hodge structure  $(E, F, L)$  of weight  $k-1$  over a Zariski open subset  $j : U \rightarrow X$ , we define an object of  $MF_{rh}(X)$  as follows. The underlying perverse sheaf is  $j_*L[1]$  which is the intersection cohomology complex for  $L$ . Since  $(E, \nabla)$  has regular singularities [Sc], we can extend it to a vector bundle with logarithmic connection on  $X$  – in many ways [De2]. The ambiguity depends on the eigenvalues of the residues of the extension which are determined mod  $\mathbb{Z}$ . For every half open interval  $I$  of length 1, there is a unique extension  $\bar{E}^I$  with eigenvalues in  $I$ . Let  $\mathcal{M}' = \bigcup_I \bar{E}^I \subseteq j_*E$ . This is a  $D_X$ -module which corresponds to the perverse sheaf  $\mathbb{R}j_*L[1] \otimes \mathbb{C}$ . Let  $\mathcal{M} \subseteq \mathcal{M}'$  be the sub  $D_X$ -module generated by  $\bar{E}^{(-1,0]}$ . This corresponds to what we want, namely  $j_*L[1]$ . We filter this by

$$F_p \bar{E}^{(-1,0]} = j_* F_p E \cap \bar{E}^{(-1,0]}$$

$$F_p \mathcal{M} = \sum_i F_i D_X F_{p-i} \bar{E}^{(-1,0]}.$$

Since the monodromy is quasi-unipotent [Sc, 4.5], the  $V$ -filtration exists for general reasons. However, in this case we can realize this explicitly by

$$V_\alpha \mathcal{M} = \bar{E}^{[-\alpha, -\alpha+1]}.$$

This is essentially how we described it in Example 2.5. From this it follows that

$$F_p \cap V_\alpha \mathcal{M} = j_* F_p E \cap \bar{E}^{[-\alpha, -\alpha+1]}$$

for  $\alpha < 1$ . The collection  $(\mathcal{M}, F_\bullet \mathcal{M}, j_*L[1])$  defines an object of  $MF_{rh}(X)$  that we call a polarizable Hodge module of type 1 of weight  $k$ . A polarizable Hodge module of weight  $k$  is a finite direct sum of objects of these two types. Let  $MH(X, k)^p$  denote the full subcategory of these.

**Theorem 4.1**  $MH(X, k)^p$  is abelian and semisimple.

In outline, the proof can be reduced to the following observations. We claim that there are no nonzero morphisms between objects of type 0 and type 1. To see this, we can replace  $X$  by a disk and assume that  $x$  and  $U$  above correspond to 0 and  $\Delta^*$  respectively. Then by Lemma 2.2, the perverse sheaves of type 0 and 1 correspond to the quivers

$$0 \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} \phi$$

$$\psi \begin{array}{c} \xrightarrow{\text{onto}} \\ \xleftarrow{1-1} \end{array} \phi$$

and the claim follows. We are thus reduced to dealing with the types separately. For type 0 (respectively 1), we immediately reduce it the corresponding statements for the categories of polarizable Hodge structures (respectively variation of Hodge structures), where it is standard. In essence the polarizations allow one to take orthogonal complements, and hence conclude semisimplicity.

**Theorem 4.2** *If  $\mathcal{M} \in MH(X, k)^p$ , then its cohomology  $H^i(\mathcal{M})$  carries a pure Hodge structure of weight  $k$ .*

**Proof** Since  $MH(X, k)^p$  is semisimple, we can assume that  $\mathcal{M}$  is simple. Then either  $\mathcal{M}$  is supported at point or it is of type 1. In the first case,  $H^0(\mathcal{M}) = \mathcal{M}$  is already a Hodge structure of weight  $k$  by definition, and the higher cohomologies vanish. In the second case, we appeal to Theorem 3.5 or just the special case due to Zucker. □

We end with a local analysis of Hodge modules. By a *Hodge quiver* we mean a diagram

$$\psi \begin{array}{c} \xrightarrow{c} \\ \xleftarrow{v} \end{array} \phi$$

where  $\psi$  and  $\phi$  are mixed Hodge structures,  $c$  is morphism and  $v$  is a morphism  $\phi \rightarrow \psi(-1)$  to the Tate twist, and such that the compositions  $c \circ v, v \circ c$  (both denoted by  $N$ ) are nilpotent. Actually, we will need to add a bit more structure, but we can hold off on this for the moment. We can encode the local structure of a Hodge module by a Hodge quiver. A module of type 0 and weight  $k$  corresponds to the quiver

$$0 \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} \phi$$

where  $\phi$  is pure of weight  $k$ .

Next, consider the type 1 modules. We wish to associate a Hodge quiver to a polarizable variation of Hodge structure  $(E, F, L)$  of weight  $k - 1$  over  $\Delta^*$ . First suppose that the monodromy is unipotent. Then let  $\psi = {}^p\psi_t L[1]$  with its limit mixed Hodge structure i.e. the mixed Hodge structure on  $\bar{E}_0^{(-1,0]}$  with Hodge filtration given by  $F^\bullet \bar{E}_0^{(-1,0]}$  and weight filtration  $W = M(N)[k - 1]$ . Set  $\phi = im(N) = \psi/ker(N)$  with its induced mixed Hodge structure. Then

$$\psi \begin{array}{c} \xrightarrow{N} \\ \xleftarrow{id} \end{array} \phi$$

gives the corresponding Hodge quiver. The Hodge quivers arising from this construction are quite special in that  $(\psi, F, N)$  and  $(\phi, F, N)$  are nilpotent orbits of weight  $k - 1$  and  $k$  respectively by [KK, 2.1.5].

In general, the monodromy is quasi-unipotent. Thus for some  $r$ , the pullback of  $L$  along  $\pi : t \mapsto t^r$  is unipotent. So we can repeat the previous construction. But now the Galois group  $\mathbb{Z}/r\mathbb{Z}$  will act on everything, and it will be necessary to include this action as part of the structure of a Hodge quiver so as not to lose any information. This can be understood directly without doing a base change. The decomposition of  $\psi = {}^p\psi_t L[1] \cong {}^p\psi_t \pi^* L$  into a sum isotypic  $\mathbb{Z}/r\mathbb{Z}$ -modules can be identified with the decomposition into generalized eigenspaces for the  $T$ -action. By Theorem 2.9, we can realize this decomposition by

$$\psi = \bigoplus_{0 \leq \alpha < 1} Gr_\alpha^V \mathcal{M}.$$

For the Hodge filtration, we use the induced filtration

$$F^\bullet Gr^V \mathcal{M} = \bigoplus_\alpha \frac{F^\bullet \cap V_\alpha \mathcal{M}}{F^\bullet \cap V_{\alpha-\epsilon} \mathcal{M}} = \bigoplus_\alpha \frac{j_* F^\bullet \cap \bar{E}^{[-\alpha, -\alpha+1]}}{j_* F^\bullet \cap \bar{E}^{(-\alpha, -\alpha+1]}}$$

where  $\mathcal{M} = D_\Delta \bar{E}^{(-1, 0]}$  is defined as above. For  $N$  we take the logarithm of the unipotent part of  $T$  under the multiplicative Jordan decomposition. Then  $(\psi, F, N)$  is a nilpotent orbit of weight  $k - 1$ , which determines a mixed Hodge structure.  $\phi$  and the rest of the diagram is defined as before. In this case, we include the grading of  $\psi, \phi$  by eigenspaces as part of the structure of a Hodge quiver. In this way, we can recover the full monodromy  $T$  rather than just  $N$ .

In summary, every polarizable Hodge module on the disk gives rise to a Hodge quiver. This yields a functor which is necessarily faithful, since the quiver determines the underlying perverse sheaf. This can be made into a full embedding by adding more structure to a Hodge quiver (namely, a choice of a variation of Hodge structure on  $\Delta^*, \dots$ ). However, this will not be necessary for our purposes.

## 4.2 Hodge modules: overview

In the next section, we will define the full subcategories  $MH(X, n) \subseteq MF_{rh}(X, \mathbb{Q})$  of *Hodge modules* of weight  $n \in \mathbb{Z}$  in general. Since this is rather technical, we start by explaining the main results. Aside from Saito's original papers [S1, S3], further explanations can be found in Brylinski-Zucker [BZ], Peters-Steenbrink [PS] and Sabbah [Sb]. The construction and basic properties of the categories are contained in [S1, §5.1-5.2]:

**Theorem 4.3 (Saito)**  *$MH(X, n)$  is abelian, and its objects possess strict support decompositions, i.e. that the maximal sub/quotient module with*

support in a given  $Z \subseteq X$  can be split off as a direct summand. There is an abelian subcategory  $MH(X, n)^P$  of polarizable objects which is semisimple.

We essentially checked these properties for polarizable Hodge modules on curves in section 4.1. They have strict support decompositions by the way we defined them. Let  $MH_Z(X, n) \subseteq MH(X, n)$  denote the subcategory of Hodge modules with strict support in  $Z$ , i.e. that all sub/quotient modules have support exactly  $Z$ . The main examples are provided by the following.

**Theorem 4.4 (Saito)** *Any weight  $m$  polarizable variation of Hodge structure  $(L, \dots)$  over an open subset of a closed subset*

$$U \xrightarrow{j} Z \xrightarrow{i} X$$

can be extended to a polarizable Hodge module in  $MH_Z(X, n)^P \subseteq MH(X, n)^P$  with  $n = m + \dim Z$ . The underlying perverse sheaf of the extension is the associated intersection cohomology complex  $i_*j^*L[\dim U]$ . All simple objects of  $MH(X, n)^P$  are of this form.

**Proof** This follows from theorem 3.21 of the second paper [S3] along with the discussion leading up to it. A sketch can also be found in [S2].  $\square$

Tate twists  $\mathcal{M} \mapsto \mathcal{M}(j)$  can be defined in this setting and are functors  $MH(X, n) \rightarrow MH(X, n - 2j)$ . Given a morphism  $f : X \rightarrow Y$  and  $L \in \text{Perv}(X)$ , we can define perverse direct images by  ${}^pRf_*^i L = {}^p\mathcal{H}^i(\mathbb{R}f_*L)$ . This operation extends to Hodge modules. If  $f$  is smooth and projective and  $(M, F, L) \in MH(X)$ , the direct image is  ${}^pRf_*^i L$  with the associated filtered  $D$ -module given by a Gauss-Manin construction:

$$(\mathbb{R}^i f_*(\Omega_{X/Y}^\bullet \otimes_{\mathcal{O}_X} M), \text{im}[\mathbb{R}^i f_*(\Omega_{X/Y}^\bullet \otimes_{\mathcal{O}_X} F_{p+\bullet} M)])$$

When applied to  $(\mathcal{O}_X, \mathbb{Q}_X[\dim X])$  with trivial filtration, we recover the variation of Hodge structure of example 3.3 (after adjusting notation).

**Theorem 4.5 (Saito [S1, 5.3.1])** *Let  $f : X \rightarrow Y$  be a projective morphism with a relatively ample line bundle  $\ell$ . If  $\mathcal{M} = (M, F, L) \in MH(X, n)$  is polarizable, then*

$${}^pRf_* \mathcal{M} \in MH(Y, n + i)$$

Moreover, a hard Lefschetz theorem holds:

$$\ell^j \cup : {}^pR^{-j} f_* \mathcal{M} \cong {}^pR^j f_* \mathcal{M}(j)$$

**Corollary 4.6** *Given a polarizable variation of Hodge structure defined on an open subset of  $X$ , its intersection cohomology carries a pure Hodge structure. This cohomology satisfies the Hard Lefschetz theorem.*

The last statement was originally obtained in the geometric case in [BBD]. The above results yield a Hodge theoretic proof of the decomposition theorem of [loc. cit.]

**Corollary 4.7** *With assumptions of the theorem  $\mathbb{R}f_*\mathbb{Q}$  decomposes into a direct sum of shifts of intersection cohomology complexes.*

### 4.3 Hodge modules: conclusion

We now give the precise definition of Hodge modules. This is given by induction on dimension of the support. This inductive process is handled via vanishing cycles. We start by explaining how to extend the construction to  $MF_{rh}(X, \mathbb{Q})$ . Given a morphism  $f : X \rightarrow \mathbb{C}$ , and a  $D_X$ -module,  $M$ , we introduced the Kashiwara-Malgrange filtration  $V$  on  $M$  in Section 2.3. Now suppose that we have a good filtration  $F$  on  $M$ . The pair  $(M, F)$  is said to be *quasi-unipotent and regular* along  $f^{-1}(0)$  if  $V$  exists and if

1.  $t(F_p V_\alpha \tilde{M}) = F_p V_{\alpha-1} \tilde{M}$  for  $\alpha < 1$ .
2.  $\partial_t(F_p Gr_\alpha^V \tilde{M}) = F_{p+1} Gr_{\alpha+1}^V \tilde{M}$  for  $\alpha \geq 0$ .

hold along with certain finiteness properties that we will not spell out. Saito [S1, 3.4.12] has shown Theorem 2.9 holds assuming only the existence of such a filtration  $F$ . As for examples, note that a variation of Hodge structure on a disk can be seen to satisfy these conditions using the formulas of Section 4.1 and [S1, 3.2.2]. Also any  $D$ -module which is quasi-unipotent and regular in the usual sense admits a filtration  $F$  as above.

We extend the functors  $\phi$  and  $\psi$  to  $MF_{rh}(X, \mathbb{Q})$  by

$$\begin{aligned} \psi_{f,e(\alpha)}(M, F, L) &= (Gr_\alpha^V(\tilde{M}), F[1], {}^p\psi_{f,e(\alpha)}L), \quad 0 \leq \alpha < 1 \\ \phi_{f,e(\alpha)}(M, F, L) &= (Gr_{\alpha+1}^V(\tilde{M}), F, {}^p\phi_{f,e(\alpha)}L), \quad 0 \leq \alpha < 1 \\ \psi_f &= \bigoplus \psi_{f,\lambda}, \quad \phi_f = \bigoplus \phi_{f,\lambda} \end{aligned}$$

Where  $F$  on the right denotes the induced filtration. Thanks to the shift in  $F$ ,  $\partial_t$  and  $t$  induce morphisms

$$\begin{aligned} \text{can} : \psi_{f,e(\alpha)}(M, F, L) &\rightarrow \phi_{f,e(\alpha)}(M, F, L) \\ \text{var} : \phi_{f,e(\alpha)}(M, F, L) &\rightarrow \psi_{f,e(\alpha)}(M, F, L) \end{aligned}$$

respectively.

We are ready to give the inductive definition of  $MH(X, n)$ . The elementary definition for curves given earlier will turn out to be equivalent. Define  $X \mapsto MH(X, n)$  to be the smallest collection of full subcategories of  $MF_{rh}(X, \mathbb{Q})$  satisfying:

- (MH1) If  $(M, F, L) \in MF_{rh}(X, \mathbb{Q})$  has zero dimensional support, then it lies in  $MH(X, n)$  iff its stalks are Hodge structures of weight  $n$ .
- (MH2) If  $(M, F, L) \in MHS(X, n)$  and  $f : U \rightarrow \mathbb{C}$  is a general morphism from a Zariski open  $U \subseteq X$ , then
  - (a)  $(M, F)|_U$  is quasi-unipotent and regular with respect to  $f$ .
  - (b)  $(M, F, L)|_U$  decomposes into a direct sum of a module supported in  $f^{-1}(0)$  and a module for which no sub or quotient module is supported in  $f^{-1}(0)$ .
  - (c) If  $W$  is the monodromy filtration of  $\psi_f(M, F, L)|_U$  (with respect to the log of the unipotent part of monodromy) centered at  $n - 1$ , then  $Gr_i^W \psi_f(M, F, L)|_U \in MH(U, i)$ . Likewise for  $Gr_i^W \phi_{f,1}(M, F, L)$  with  $W$  centered at  $n$ .

This is a lot to absorb, so let us make few remarks about the definition.

- If  $f^{-1}(0)$  is in general position with respect to  $\text{supp } M$ , the dimension of the support drops after applying the functors  $\phi$  and  $\psi$ . Thus this is an inductive definition.
- The somewhat technical condition (b) ensures that Hodge modules admit strict support decompositions. The condition can be rephrased as saying that  $(M, F, L)$  splits as a sum of the image of  $\text{can}$  and the kernel of  $\text{var}$ . A refinement of Lemma 2.4 shows that  $L$  will then decompose into a direct sum of intersection cohomology complexes. This is ultimately needed to be able to invoke Theorem 3.5 when the time comes to construct a Hodge structure on cohomology.
- Although  $MF_{rh}(X, \mathbb{Q})$  is not an abelian category, the category of compatible pairs consisting of a  $D$ -module and perverse sheaf is. Thus we do get a  $W$  filtration for  $\psi_f(M, F, L)|_U$  in (c) by Proposition 3.7 by first suppressing  $F$ , and then using the induced filtration.

There is a notion of polarization in this setting. Given  $(M, F, L) \in MH_Z(X, n)$ , a polarization is a pairing  $S : L \otimes L \rightarrow \mathbb{Q}_X[2 \dim X](-n)$  satisfying certain axioms. The key conditions are again inductive. When  $Z$  is a point,  $S$  should correspond to a polarization on the Hodge structure at the stalk in the usual sense. In general, given a (germ of a) function  $f : Z \rightarrow \mathbb{C}$  which is not identically zero,  $S$  should induce a polarization on

the nearby cycles  $Gr_{\bullet}^W \psi_f L$  (using the same recipe as in Theorem 3.9 (3)). Once all the definitions are in place, the proofs of the theorems involve a rather elaborate induction on dimension of supports.

In order to get a better sense of what Hodge modules look like, we extend the local description given in §4.1 to the polydisk  $\Delta^2$ . We assume that the underlying perverse sheaves satisfying the assumptions of theorem 2.3. Then using a vanishing cycle construction refining the one used earlier, such a polarizable Hodge module gives rise to a two dimensional Hodge quiver. This is a commutative diagram of  $\mathbb{Q}^2$ -graded mixed Hodge structures

$$\begin{array}{ccc} H_{11} & \begin{array}{c} \xleftarrow{c} \\ \xrightarrow{v} \end{array} & H_{12} \\ \begin{array}{c} \uparrow \\ \downarrow \end{array} \begin{array}{c} \\ c \end{array} & & \begin{array}{c} \uparrow \\ \downarrow \end{array} \begin{array}{c} \\ c \end{array} \\ H_{21} & \begin{array}{c} \xleftarrow{c} \\ \xrightarrow{v} \end{array} & H_{22} \end{array}$$

such that the maps labeled  $c$  and  $v$  are required to satisfy the same conditions as in §4.1. In particular, compositions  $c \circ v$ ,  $v \circ v$  are nilpotent. These nilpotent transformations are denoted by  $N_1$  for horizontal arrows, and  $N_2$  for vertical.

Given a Hodge module  $M$  of weight  $k$ ,  $H_{11} = \psi_{t_2} \psi_{t_1} M \in MF_{rh}(pt, \mathbb{Q})$  etcetera are filtered vector spaces with a  $\mathbb{Q}^2$ -grading by eigenspaces. These filtrations are used as the Hodge filtrations. The weight filtration is given by the shifted monodromy filtration with respect to  $N = N_1 + N_2$ , where  $N_i$  are the logarithms of the unipotent parts of monodromy around the axes. The shifts are indicated in the diagram below

$$\begin{array}{ccc} k-2 & \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} & k-1 \\ \begin{array}{c} \uparrow \\ \downarrow \end{array} & & \begin{array}{c} \uparrow \\ \downarrow \end{array} \\ k-1 & \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} & k \end{array} \quad (4.1)$$

These quivers are quite special. A Hodge quiver is pure of weight  $k$  if the conditions of Lemma 2.4 hold, and if each  $(H_{ij}, F, W, N_1, N_2)$  arises from a pure nilpotent orbit of weight indicated indicated in (4.1). With this description in hand, we can verify a local version of Theorem 4.4. Given a polarizable variation of Hodge structure  $L$  of weight  $k-2$  on  $(\Delta - \{0\})^2$  corresponding to a nilpotent orbit  $(H, F, N_1, N_2)$ , we can construct the Hodge quiver

$$\begin{array}{ccc} H & \begin{array}{c} \xleftarrow{N_1} \\ \xrightarrow{id} \end{array} & N_1 H \\ \begin{array}{c} \uparrow \\ \downarrow \end{array} \begin{array}{c} \\ N_2 \end{array} & & \begin{array}{c} \uparrow \\ \downarrow \end{array} \begin{array}{c} \\ N_2 \end{array} \\ N_2 H & \begin{array}{c} \xleftarrow{N_1} \\ \xrightarrow{id} \end{array} & N_1 N_2 H \end{array}$$

where  $H$  is equipped with the monodromy filtration  $W(N_1 + N_2)$  on the upper left. The remaining vertices are given the mixed Hodge structures induced on the images of this. The fact that these vertices correspond to nilpotent orbits of the correct weight follows from [CKS, 1.16] or [KK, 2.1.5]. Note that  $j_{!*}L[2]$  is the perverse sheaf corresponding to this diagram. The details can be found in [S3, §3].

#### 4.4 Mixed Hodge modules

Saito has given an extension of the previous theory by defining the notion of mixed Hodge module. We will define a *pre-mixed Hodge module* on  $X$  to consist of

- A perverse sheaf  $L$  defined over  $\mathbb{Q}$ , together with a filtration  $W$  of  $L$  by perverse subsheaves.
- A regular holonomic  $D_X$ -module  $\mathcal{M}$  with a filtration  $W\mathcal{M}$  which corresponds to  $(L \otimes \mathbb{C}, W \otimes \mathbb{C})$  under Riemann-Hilbert.
- A good filtration  $F$  on  $\mathcal{M}$ .

These objects form a category, and Saito defines the subcategory of *mixed Hodge modules*  $MHM(X)$  by a rather delicate induction. The key points are that for  $(\mathcal{M}, F, L, W)$  to be in  $MHM(X)$ , he requires

- The associated graded objects  $Gr_k^W(\mathcal{M}, F, L)$  yield polarizable Hodge modules of weight  $k$ .
- For any (germ of a) function  $f$  on  $X$ , the relative monodromy filtration  $M$  (resp.  $M'$ ) for  $\psi_f(\mathcal{M}, F, L)$  (resp.  $\phi_{f,1}(\mathcal{M}, F, L)$ ) with respect to  $W$  exists.
- The pre-mixed Hodge modules

$$(\psi_f(\mathcal{M}, F, L), M) \text{ and } (\phi_{f,1}(\mathcal{M}, F, L), M')$$

are in fact mixed Hodge modules on  $f^{-1}(0)$ .

This is not a complete list of all the conditions. Saito also requires that mixed Hodge modules should be extendible across divisors in a sense we will not attempt to make precise. (To avoid confusion, we are using the definition of  $MHM$  given in section 4 of [S3], where polarizability is assumed.) The main properties are summarized below:

**Theorem 4.8 (Saito)**

1.  $MHM(X)$  is abelian, and it contains each  $MH(X, n)^p$  as a full abelian subcategory.
2.  $MHM(\text{point})$  is the category of polarizable mixed Hodge structures.
3. If  $U \subseteq X$  is open, then any admissible variation of mixed Hodge structure on  $U$  extends to an object in  $MHM(X)$ .

**Proof** For the first statement see [S3, 4.2.3], and [S3, 3.9,3.27] for the remaining statements. □

Finally, we have:

**Theorem 4.9 (Saito [S3, thm 0.1])** *There is a realization functor of triangulated categories*

$$real : D^b MHM(X) \rightarrow D_{constr}^b(X, \mathbb{Q})$$

and functors

$$\begin{aligned} {}^p\mathcal{H}^i &: D^b MHM(X) \rightarrow MHM(X) \\ \mathbb{R}f_* &: D^b MHM(X) \rightarrow D^b MHM(Y) \\ \mathbb{L}f^* &: D^b MHM(Y) \rightarrow D^b MHM(X) \end{aligned}$$

for each morphism  $f : X \rightarrow Y$ , such that

$$\begin{aligned} {}^p\mathcal{H}^i \circ real &= real \circ {}^p\mathcal{H}^i \\ real(\mathbb{R}f_* \mathcal{M}) &= \mathbb{R}f_* real(\mathcal{M}) \\ real(\mathbb{L}f^* \mathcal{N}) &= \mathbb{L}f^* real(\mathcal{M}) \end{aligned}$$

*Similar statements hold for various other standard operations such as tensor products and cohomology with proper support.*

Putting these results together yields Saito's mixed Hodge structure:

**Theorem 4.10** *The cohomology of a smooth variety  $U$  with coefficients in an admissible variation of mixed Hodge structure  $L$  carries a canonical mixed Hodge structure.*

When the base is a curve, this was first proved by Steenbrink and Zucker [SZ].

**Proof**  $L$  gives an object of  $MHM(U)$ . Thus  $H^i(\mathbb{R}p_*L)$  carries a mixed Hodge structure, where  $p : U \rightarrow pt$  is the projection to a point.  $\square$

We extend the local description of perverse sheaves and Hodge modules to mixed Hodge modules. A one dimensional *filtered Hodge quiver* is a diagram

$$\psi \begin{array}{c} \xrightarrow{c} \\ \xleftarrow{v} \end{array} \phi$$

where  $\psi$  and  $\phi$  are filtered  $\mathbb{Q}$ -graded mixed Hodge structures,  $c$  and  $v : \phi \rightarrow \psi(-1)$  are morphisms and the compositions  $c \circ v, v \circ c$  (both denoted by  $N$ ) are nilpotent. So each object  $\psi, \phi$  is equipped with 3 filtrations  $F, M, W$ , where  $M$  denotes the weight filtration. For a filtered Hodge quiver to arise from a mixed Hodge module, we require that

- $Gr_W^k$  is pure of weight  $k$ .
- $(\psi, F, W, N)$  and  $(\phi, F, W, N)$  are mixed nilpotent orbits with  $M$  the relative monodromy filtration.

We call these *admissibility* conditions. This extends to polydisks in a straightforward manner.

## 4.5 Explicit construction

We will give a bit more detail on the construction of the mixed Hodge structure in Theorem 4.10. For a different perspective see [E]. Let  $U$  be a smooth  $n$  dimensional variety. We can choose a smooth compactification  $j : U \rightarrow X$  such that  $D = X - U$  is a divisor with normal crossings. Fix an admissible variation of mixed Hodge structure  $(L, W, E, F, \nabla)$  on  $U$ . Extend  $(E, \nabla)$  to a vector bundle  $\bar{E}^I$  with a logarithmic connection such that the eigenvalues of its residues lie in  $I$  as in Section 4.1 Let  $\bar{E} = \bar{E}^{(-1,0]}$  and  $\mathcal{M} = \bigcup_I \bar{E}^I \subseteq j_*E$ . Then  $\mathcal{M}$  is a  $D_X$ -module which corresponds to the perverse sheaf  $\mathbb{R}j_*L[n] \otimes \mathbb{C}$ . Filter this by

$$F_p\mathcal{M} = \sum_i F_i D_X F_{p-i} \bar{E}$$

Then:

**Theorem 4.11** *There exists compatible filtrations  $\tilde{W}$  on  $\mathbb{R}j_*L[n]$  and  $\mathcal{M}$  extending  $W$  over  $U$  such that  $(\mathbb{R}j_*L[n], \tilde{W}, \mathcal{M}, F)$  becomes a mixed Hodge module.*

We describe  $\tilde{W}$  when  $n = \dim X \leq 2$ . We start with the case of  $n = 1$ , which is essentially due to Steenbrink and Zucker [SZ]. It suffices to construct  $\tilde{W}$  locally near any point of  $p \in D$ . Choose a coordinate disk  $\Delta$  at  $p$ . The perverse sheaf  $\mathbb{R}j_*L[1]$  corresponds to the quiver

$$\psi \begin{array}{c} \xrightarrow{N} \\ \xleftarrow{id} \end{array} \psi \quad (4.2)$$

where  $\psi = \psi_t L$  and  $N$  is the logarithm of unipotent part of monodromy at  $p$ . We make this into an admissible filtered Hodge quiver by replacing  $\psi$  on the left with the mixed nilpotent orbit  $(H, F^\bullet H, W_\bullet H, N)$ , where we can take  $H = \bar{E}_0$  when the monodromy is unipotent, and  $H = \oplus Gr_\alpha^V(D_\Delta \bar{E})$  in general. On the right, we keep  $(H, F^\bullet H, N)$  but replace  $W$  with

$$(N_* W)_{k-1} := NW_k + M_k \cap W_{k-1} \quad (4.3)$$

where  $M$  is the relative monodromy filtration of  $(H, W)$ . This is a mixed nilpotent orbit by [K1, 5.5.4]. Then define the filtration  $\tilde{W}_k$  of (4.2) by

$$W_{k-1} \begin{array}{c} \xrightarrow{N} \\ \xleftarrow{id} \end{array} (N_* W)_{k-2}$$

The key point is that  $Gr_k^{\tilde{W}}$  is pure of weight  $k$ , because it is a sum of

$$Gr_{k-1}^W \begin{array}{c} \xrightarrow{N} \\ \xleftarrow{id} \end{array} NGr_{k-1}^W$$

and

$$0 \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} Gr_{k-2}^M(W_{k-2}/NW_{k-2})(-1)$$

both pure of weight  $k$ . A proof can be extracted from [SZ, pp 514-516] or [K1]. Since we have an equivalence of categories between perverse sheaves on  $\Delta$  and quivers,  $\tilde{W}$  above determines a filtration of  $\mathbb{R}j_*L[1]$  by perverse subsheaves.

We now turn to the surface case. Replace  $X$  by a polydisk  $\Delta^2$  such that  $D$  corresponds to (a subdivisor of) the union of axes. An admissible filtered Hodge quiver is now a commutative diagram

$$\begin{array}{ccc} H_{11} & \begin{array}{c} \xrightarrow{c} \\ \xleftarrow{v} \end{array} & H_{12} \\ \begin{array}{c} \uparrow v \\ \downarrow c \end{array} & & \begin{array}{c} \uparrow v \\ \downarrow c \end{array} \\ H_{21} & \begin{array}{c} \xrightarrow{c} \\ \xleftarrow{v} \end{array} & H_{22} \end{array}$$

where  $H_{ij}$  arise from mixed nilpotent orbits. The maps labeled  $c$  and  $v$  are required to satisfy the same conditions as above. Let  $N_i$  denote the logarithms of the unipotent parts of local monodromies of  $L$  around the axes. Then the perverse sheaf  $\mathbb{R}j_*L[2]$  is represented by the quiver

$$\begin{array}{ccc} H & \xrightleftharpoons[N_1]{id} & H \\ id \uparrow \downarrow & & id \uparrow \downarrow \\ H & \xrightleftharpoons[N_1]{N_2} & H \end{array}$$

where  $H = \bar{E}_0$  in the unipotent case. We make this into a filtered Hodge quiver by equipping this with  $F^\bullet H, N_i$  and the weight filtrations given below

$$\begin{array}{ccc} W_k & \xrightleftharpoons{N_1} & (N_{1*}W)_{k-1} \\ \uparrow \downarrow & & \uparrow \downarrow \\ (N_{2*}W)_{k-1} & \xrightleftharpoons{N_1} & (N_{1*}N_{2*}W)_{k-2} \end{array}$$

where  $N_{1*}W\dots$  are defined as in (4.3). Note that symmetry holds for the last filtration  $N_{1*}N_{2*}W = N_{2*}N_{1*}W$  by [K1, 5.5.5]. Then  $\tilde{W}_k = W_{k-2}$  gives the desired filtration on  $\mathbb{R}j_*L[2]$ . Again arguments of [K1] can be used to verify that  $Gr_k^{\tilde{W}}$  is pure of weight  $k$ , as required.

Finally, let us describe Saito's mixed Hodge structure explicitly by elaborating on a remark in [S2]. We have an isomorphism

$$\alpha : \mathbb{R}j_*L \otimes \mathbb{C} \cong \Omega_X^\bullet(\log D) \otimes \bar{E}^{[0,1]}$$

and the filtrations defining the Hodge structure can be displayed rather explicitly using the complex on the right. We define two filtrations  $F$  and  $\tilde{W}$  on this complex.

$$F^p(\Omega_X^\bullet(\log D) \otimes \bar{E}^{[0,1]}) = \bigoplus_i \Omega_X^i(\log D) \otimes F^{p-i} \bar{E}^{[0,1]}$$

We construct  $\tilde{W}$  by taking the filtration induced by  $(\Omega_X^\bullet \otimes \tilde{W}_{\bullet+n}\mathcal{M})[n]$  under the inclusion

$$\Omega_X^\bullet(\log D) \otimes \bar{E}^{[0,1]} \subset \Omega_X^\bullet \otimes \mathcal{M}[n]$$

We recall at this point that Deligne introduced a device, called a *cohomological mixed Hodge complex* [De3, §8.1], for producing mixed Hodge structures. This consists of bifiltered complex  $(A_{\mathbb{C}}, W_{\mathbb{C}}, F)$  of sheaves of  $\mathbb{C}$ -vector spaces, a filtered complex of  $(A, W)$  of sheaves over  $\mathbb{Q}$ , and a filtered quasi-isomorphism  $(A, W) \otimes \mathbb{C} \cong (A_{\mathbb{C}}, W_{\mathbb{C}})$ . The crucial axioms are that

- this datum should induce a pure weight  $i + k$  Hodge structure on  $H^i(Gr_k^W A)$
- the filtration induced by  $F$  on  $Gr_k^W$  is strict, i.e. the map

$$H^i(F^p Gr_k^W A) \rightarrow H^i(Gr_k^W A)$$

is injective.

**Proposition 4.12** *With these filtrations*

$$(\mathbb{R}j_* L, \tilde{W}; \Omega_X(\log D)^\bullet \otimes \bar{E}, \tilde{W}, F; \alpha)$$

*becomes a cohomological mixed Hodge complex.*

To verify the above axioms, one observes that  $Gr_k^{\tilde{W}} \mathbb{R}j_* L$  decomposes into a direct sum of intersection cohomology complexes associated to pure variations of Hodge structure of the correct weight. An appeal to Theorem 3.5 shows that  $H^i(Gr_k^{\tilde{W}} \mathbb{R}j_* L)$  carries pure Hodge structures of weight  $k + i$ . Strictness follows from similar considerations.

From [De3, §8.1], we obtain

**Corollary 4.13** *The Hodge filtration is induced by  $F$  under the isomorphism*

$$H^i(U, L \otimes \mathbb{C}) \cong \mathbb{H}^i(\Omega_X^\bullet(\log D) \otimes \bar{E}^{[0,1]})$$

*The weight filtration is given by*

$$W_{i+k} H^i(U, L \otimes \mathbb{C}) = \mathbb{H}^i(X, W_k(\Omega_X^\bullet(\log D) \otimes \bar{E}^{[0,1]})).$$

Finally, we remark that in the simplest case  $L = \mathbb{Q}$ ,  $\tilde{W}$  above coincides with the filtration  $W$  defined by Deligne [De3, §3.1]; in particular, Saito's Hodge structure coincides with Deligne's in this case.

## 5 Comparison

From now on let  $f : X \rightarrow Y$  be a smooth projective morphism of smooth quasiprojective varieties. We recall two results

**Theorem 5.1 (Deligne [De1])** *The Leray spectral sequence*

$$E_2^{pq} = H^p(Y, R^q f_* \mathbb{Q}) \Rightarrow H^{p+q}(X, \mathbb{Q})$$

*degenerates. In particular  $E_2^{pq} \cong Gr_L^p H^{p+q}(X)$  for the associated "Leray filtration"  $L$ .*

**Theorem 5.2 ([A])** *There exists varieties  $Y_p$  and morphisms  $Y_p \rightarrow Y$  such that*

$$L^p H^i(X, \mathbb{Q}) = \ker[H^i(X, \mathbb{Q}) \rightarrow H^i(X_p, \mathbb{Q})]$$

where  $X_p = f^{-1}Y_p$ .

It follows that each  $L^p$  is a filtration by sub mixed Hodge structures. When combined with Theorem 5.1, we get a mixed Hodge structure on  $H^p(Y, R^q f_* \mathbb{Q})$  which we will call the naive mixed Hodge structure. On the other hand,  $R^q f_* \mathbb{Q}$  carries a pure hence admissible variation of Hodge structure, so we can apply Saito's Theorem 4.10.

**Theorem 5.3** *The naive mixed Hodge structure on  $H^p(Y, R^q f_* \mathbb{Q})$  coincides with Saito's.*

**Proof** Note that  $R^i f_* \mathbb{Q}$  are local systems and hence perverse sheaves up to shift. More specifically,

$$R^i f_* \mathbb{Q} = \text{real}({}^p \mathcal{H}^{i-\dim X + \dim Y} \mathbb{R} f_* \mathbb{Q}[\dim X])[-\dim Y]$$

Deligne [De1] actually proved a stronger version of the above theorem which implies that

$$\mathbb{R} f_* \mathbb{Q} \cong \bigoplus_i R^i f_* \mathbb{Q}[-i] \quad (5.1)$$

(non canonically) in  $D_{\text{const}}^b(X)$ . By Theorem 4.5 and [loc. cit.], we have the corresponding decomposition

$$\mathbb{R} f_* \mathbb{Q}[\dim X] = \bigoplus_i {}^p \mathcal{H}^i(\mathbb{R} f_* \mathbb{Q}[\dim X])[-i]$$

in  $D^b \text{MHM}(Y)$ . Note that the Leray filtration is induced by the truncation filtration

$$L^p H^i(X, \mathbb{Q}) = \text{image}[H^i(Y, \tau_{\leq i-p} \mathbb{R} f_* \mathbb{Q}) \rightarrow H^i(Y, \mathbb{R} f_* \mathbb{Q})]$$

Under (5.1),

$$\tau_{\leq p} \mathbb{R} f_* \mathbb{Q} \cong \bigoplus_{i \leq p} R^i f_* \mathbb{Q}[-i]$$

Therefore we have an isomorphism of mixed Hodge structures

$$H^i(X, \mathbb{Q}) \cong \bigoplus_{p+q=i} H^p(Y, R^q f_* \mathbb{Q})$$

where the right side is equipped with Saito's mixed Hodge structure. Under this isomorphism,  $L^p$  maps to

$$H^{i-p}(R^p f_* \mathbb{Q}) \oplus H^{i-p+1}(R^{p-1} f_* \mathbb{Q}) \oplus \dots$$

The theorem now follows. □

As pointed out by the referee the theorem was also proved in [PS, 14.1.3] using compatibility of the perverse Leray spectral sequence with  $MHM$ . It can also be deduced from the decomposition theorem. A somewhat stronger statement would also follow from the remark [S3, 4.6.2] that one can build a  $t$ -structure on  $D^b MHM(X)$  which agrees with the standard one on  $D_{const}^b(X)$ . However, we decided not to include this since a detailed justification for the remark was not given.

## References

- [A] D. Arapura, *The Leray spectral sequence is motivic*, Invent. Math. **160** (2005), no. 3, 567–589.
- [BBD] J. Bernstein, A. Beilinson, P. Deligne, *Faisceaux perverse*, in *Analysis and topology on singular spaces, I (Luminy, 1981)*, 5–171, Astérisque **100**, Soc. Math. France, Paris, 1982.
- [Bo] A. Borel, P.-P. Grivel, B. Kaup, A. Haefliger, B. Malgrange, F. Ehlers, *Algebraic D-modules*, Perspectives in Mathematics, 2, Academic Press, 1987.
- [Br] J.-L. Brylinski, *(Co)-homologie intersection et faisceaux pervers*, in *Bourbaki Seminar*, Vol. 1981/1982, pp. 129–157, Astérisque 92–93, Soc. Math. France, Paris, 1982.
- [BZ] J.-L. Brylinski and S. Zucker, *An overview of recent advances in Hodge theory*, in *Several complex variables, VI*, 39–142, Encyclopaedia Math. Sci., **69**, Springer, 1990.
- [CK] E. Cattani and A. Kaplan, *Polarized mixed Hodge structures and the local monodromy of a variation of Hodge structure*, Invent. Math. **67** (1982), no. 1, 101–115.
- [CKS] E. Cattani, A. Kaplan and W. Schmid,  *$L^2$  and intersection cohomologies for a polarizable variation of Hodge structure*, Invent. Math. **87** (1987), no. 2, 217–252.

- [De1] P. Deligne, *Théorème de Lefschetz et critères de dégénérescence de suites spectrales*, Inst. Hautes Études Sci. Publ. Math. **35** (1968), 259–278.
- [De2] P. Deligne, *Équation différentielles à points singuliers régulier*, Lecture Notes in Mathematics **163**, Springer-Verlag, 1970.
- [De3] P. Deligne, *Théorie de Hodge II, III*, Inst. Hautes Études Sci. Publ. Math. **40**, (1971), 5–57; **44** (1974), 5–77.
- [SGA7] P. Deligne and N. Katz, *SGA7 II*, Springer-Verlag (1973).
- [E] F. El Zein, *Deligne-Hodge-De Rham theory with coefficients*, ArXiv preprint math.AG/0702083.
- [GGM] A. Galligo, M. Granger and Ph. Maisonobe, *D-modules et faisceaux pervers dont le support singulier est un croisement normal*, Ann. Inst. Fourier (Grenoble) **35** (1985), no. 1, 1–48.
- [GoM] M. Goresky, R. Macpherson, *Intersection homology II*, Invent Math. (1983)
- [HTT] R. Hotta, K. Takeuchi and T. Tanisaki, *D-modules, Perverse sheaves, and representation theory*, Birkhäuser, 2008.
- [K1] M. Kashiwara, *A study of a variation of mixed Hodge structure*, Publ. Res. Inst. Math. Sci. **22** (1986), no. 5, 991–1024.
- [K2] M. Kashiwara, *D-modules and microlocal calculus*, Translations of Mathematical Monographs, **217**, Iwanami Series in Modern Mathematics. American Mathematical Society, 2003.
- [KK] M. Kashiwara and T. Kawai, *Poincaré lemma for variations of polarised Hodge structure*, Publ. Res. Inst. Math. Sci. **23** (1987), no. 2, 345–407.
- [PS] C. Peters and J. Steenbrink, *Mixed Hodge structures*, Results in Mathematics and Related Areas, 3rd Series, A Series of Modern Surveys in Mathematics **52**, Springer-Verlag, 2008.
- [Sb] C. Sabbah, *Hodge theory, singularities and D-modules*, Notes available from <http://www.math.polytechnique.fr/~sabbah>.
- [S1] M. Saito, *Modules de Hodge polarizables*, Publ. Res. Inst. Math. Sci. **24** (1988), no. 6, 849–995.

- [S2] M. Saito, *Mixed Hodge modules and admissible variations*, C.R. Acad. Sci. Sér. I Math. **309** (1989), no. 6, 351–356.
- [S3] M. Saito, *Mixed Hodge modules*, Publ. Res. Inst. Math. Sci. **26** (1990), no. 2, 221–333.
- [Sc] W. Schmid, *Variation of Hodge structure: singularities of the period mapping*, Invent. Math. **22** (1973), 211–319.
- [SZ] J. Steenbrink and S. Zucker, *Variations of mixed Hodge structure I*, Invent. Math. **80** (1983), no. 3, 489–542.
- [V] J-L. Verdier, *Extension of a perverse sheaf over a closed subspace*, Astérisque **130** (1985), 210–217.
- [Z] S. Zucker, *Hodge theory with degenerating coefficients*, Ann. of Math. (2) **109** (1979), no. 3, 415–476.

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