Steenrod cup-i products: axioms and algorithms

In 1947, Steenrod introduced by means of formulae the cup-i products on the cochains of standard simplices. These bilinear maps give rise on any space X to the natural cohomology operations

$$\mathrm{Sq}^k : H^\bullet(X; \mathbb{F}_2) \to H^{\bullet+k}(X; \mathbb{F}_2)$$

at the heart of stable homotopy theory.

Steenrod's formulae for the cup-i products extend the Alexander-Whitney product on cochains. This non-commutative product descends to the commutative cup product in cohomology, and we can interpret the higher cup-i products as coherent homotopies enforcing the derived commutativity at the cochain level.

In this talk we describe four axioms that characterize Steenrod's original cup-i products up to isomorphism, remarking that, to the author's knowledge, all alternative constructions in the literature satisfy them as well. Additionally, we present new formulas for these operations from which we obtain fast algorithms for the computation of Steenrod squares Sq^k and cup-i products of finite simplicial complexes.