Quadratic functions

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MA 158 Lesson 10

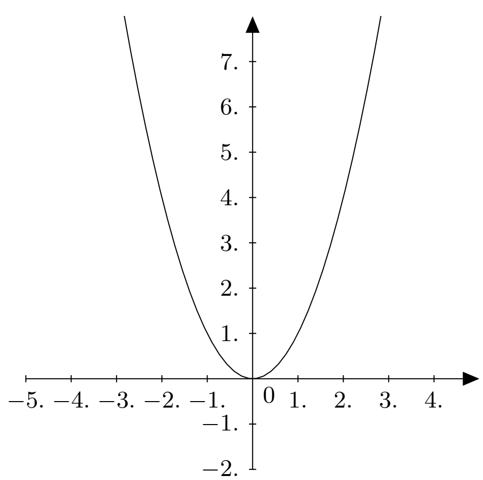
Standard form

Definition 1. A *quadratic function* is one of the form

$$y = ax^2 + bx + c,$$

where $a \neq 0$.

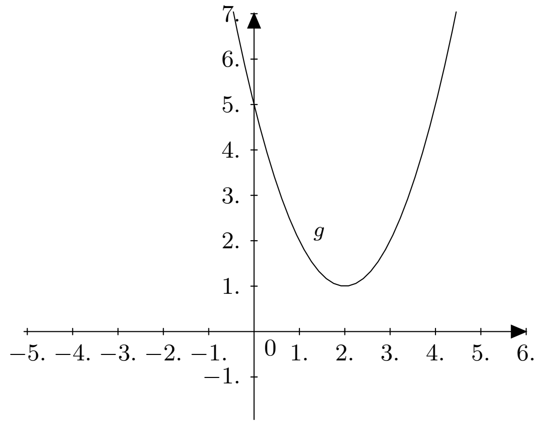
The simplest example is $f(x) = x^2$:



We can then ask ourselves what happens when we modify this a bit, say

$$g(x) = (x - 2)^2 + 1.$$

This corresponds to a translation of our original graph $f(x) = x^2$. If we add 2 to every x-value and 1 to every y-value, we end up back where we started.



Notice that the vertex of this parabola is at the point $(2, 1)$. Just for fun, it's also easy to find out where the y-intercept is. From the graph, we clearly see that it is 5, but we can also find $g(0)$:

$$\begin{aligned} g(0) &= (0-2)^2 + 1 \\ &= (-2)^2 + 1 \\ &= 4 + 1 = 5. \end{aligned}$$

Finding the vertex so easily was no coincidence. An equation for a parabola written in this form is called standard form.

Definition 2. The *standard form* for a parabola is

$$f(x) = a(x-h)^2 + k,$$

where (h, k) is the vertex of the parabola.

Remark. When $a > 0$, then the parabola opens upward ("up like a cup"), and when $a < 0$, the parabola opens downward ("down like a frown").

Example 1. Graph, find the x- and y-intercepts, and find the vertex of $f(x) = -2(x+3)^2 + 4$.

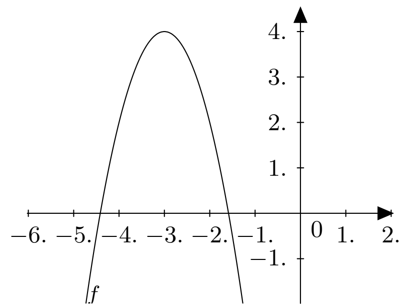
Solution. Since this parabola is written in standard form, it should be readily seen that the vertex is $(-3, 4)$. (Notice that the sign of "h" is always the opposite.) To find the y-intercept, plug in 0 for x :

$$f(0) = -2(0+3)^2 + 4 = -2(9) + 4 = -14.$$

To find the x-intercepts, we set $f(x) = 0$:

$$\begin{aligned} -2(x+3)^2 + 4 &= 0 \\ -2(x+3)^2 &= -4 \\ (x+3)^2 &= 2 \\ x + 3 &= \pm\sqrt{2} \\ x &= -3 \pm \sqrt{2}. \end{aligned}$$

And finally the graph:



□

Remark. It may help you graph to pick out a few x-values and make a T-table with the y-values to plot some additional points on the graph. Using f from the previous example,

x	$f(x)$
-5	-4
-4	2
-2	2
-1	-4

This should be enough to get the general picture. Our graphs don't need to be terribly precise, but they should be detailed enough to be helpful. Otherwise, there's no point in making the graph to begin with.

General form

We defined a parabola as something of the form ax^2+bx+c, but then we started talking about parabolas of an entirely different form. This ax^2+bx+c is called *general form*, and we can always obtain a parabola in general form starting in standard form as follows:

$$\begin{aligned} f(x) &= a(x-h)^2 + k \\ &= a(x^2 - 2hx + h^2) + k \\ &= ax^2 - 2ahx + ah^2 + k. \end{aligned}$$

Then if we call $b = -2ah,\ c = ah^2 + k$, then we have a parabola in general form.

Notice that the a from standard form is the coefficient of the x^2 term in general form. So what do we like about general form? Standard form was great for finding the vertex, but general form makes it easy to find the y-intercept and x-intercepts. To find the y-intercept, we plug in 0 for x, but that just leaves us with the c term.

To find the x-intercepts, we can do one of two things:

1. Factor the quadratic as a product of linear functions

2. Use the quadratic formula.

Reacall. The quadratic formula is used to find the *zeros*, or x-intercepts of a parabola:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

For a fun way to memorize this formula, you can check out my high school precal teacher's song on Youtube: `https://www.youtube.com/watch?v=FmBL6C8Px38`.

Notice. There is a vertical line of symmetry about the vertex. So from the quadratic formula, we can discern that the x-value of the vertex is $\frac{-b}{2a}$. So the vertex of a parabola in general form can be found by computing

$$\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right).$$

As a final remark on the vertex, it should be pretty clear by the graphs you have seen that the vertex is always a maximum or a minimum. If $a < 0$ then it will be a maximum, and if $a > 0$ it will be a minimum. At first this may seem counterintuitive, but remember $a < 0$ means it opens "up like a cup", so all of the y-values will be larger than the y-value of the vertex.

General form to standard form

The conversion from standard form to general form was pretty easy. To go the opposite direction will require a little more work. For this, we need to know the process of "completing the square." So we first consider a parabola in general form whose a term is 1: $y = x^2 + bx + c$. Notice that

$$\begin{aligned}\left(x + \frac{1}{2}b\right)^2 &= x^2 + \frac{1}{2}bx + \frac{1}{2}bx + \frac{b^2}{4} \\ &= x^2 + bx + \frac{b^2}{4},\end{aligned}$$

so if we add and subtract $\frac{b^2}{4}$, we can write

$$\begin{aligned}y &= x^2 + bx + \frac{b^2}{4} + c - \frac{b^2}{4} \\ &= \left(x + \frac{1}{2}b\right)^2 + c - \frac{b^2}{4}.\end{aligned}$$

If $a \neq 1$, then we factor out a and proceed as before. So here is the general procedure

1. Make sure the a term is 1. If not, factor out a to get $a\left(x^2 + \frac{b}{a}x\right) + c$
2. Add and subtract $\frac{b^2}{4}$. If you have factored out an a term in step 1, be sure you do this inside the parentheses.

3. Put it all together:

$$\begin{aligned} ax^2 + bx + c &= a\left(x^2 + \frac{b}{a}x\right) + c \\ &= a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right) + c \\ &= a\left(\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right) + c \\ &= a\left(x + \frac{b}{2a}\right)^2 - a\left(\frac{b}{2a}\right)^2 + c \end{aligned}$$

This may seem like utter nonsense with all the a's, b's and c's. So lets see a couple concrete examples.

Example 2. Convert to standard form and find the vertex and state whether the vertex is a maximum or a minimum.

$$y = 2x^2 - 12x - 37$$

Solution. Since $a = 2 > 0$, we can already tell that the vertex is a minimum.

$$\begin{aligned} y &= 2x^2 - 12x - 37 \\ &= 2(x^2 - 6x) - 37 \\ &= 2\left(x^2 - 6x + \left(\frac{-6}{2}\right)^2 - \left(\frac{-6}{2}\right)^2\right) - 37 \\ &= 2(x^2 - 6x + 9 - 9) - 37 \\ &= 2\left((x-3)^2 - 9\right) - 37 \\ &= 2(x-3)^2 - 18 - 37 \\ &= 2(x-3)^2 - 55. \end{aligned}$$

Now it is easy to see that the vertex is $(3, -55)$. □

Example 3. The same instructions for the polynomial $y = -4x^2 - 8x + 2$.

Solution. Here $a = -4 < 0$, so the vertex will be a maximum. Now

$$\begin{aligned} y &= -4x^2 - 8x + 2 \\ &= -4(x^2 - 2x) + 2 \\ &= -4(x^2 - 2x + 1 - 1) + 2 \\ &= -4\left((x-1)^2 - 1\right) + 2 \\ &= -4(x-1)^2 + 6, \end{aligned}$$

and the vertex is at $(1, 6)$. □

More on the quadratic formula

For a parabola in standard form ($y = ax^2 + bx + c$), the *discriminant* is the stuff under the radical in the quadratic formula. We will denote this $D = b^2 - 4ac$. We could then rewrite the quadratic formula as

$$\frac{-b \pm \sqrt{D}}{2a}.$$

Now there are 3 possibilities:

1. $D < 0$: then we would be taking the square root of a negative number, which has no real solutions. This amounts to having no x-intercepts.
2. $D = 0$: there is exactly one solution. In this case there is only one x-intercept. (How is that possible?)
3. $D > 0$: then there are two real solutions, which amounts to having two x-intercepts.

Example 4. Find the vertex, zeros, and y-intercept of $y = -3x^2 + 10x + 6$.

Solution. Recall that the x-value of the vertex is $\frac{-b}{2a}$. So the vertex is at the point

$$\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right) = \left(\frac{-10}{2(-3)}, f\left(\frac{-10}{2(-3)}\right)\right) = \left(\frac{5}{3}, \frac{43}{3}\right).$$

Using the quadratic formula to find the zeros,

$$\begin{aligned} x &= \frac{-10 \pm \sqrt{10^2 - 4(-3)(6)}}{2(-3)} = \frac{-10 \pm \sqrt{172}}{-6} \\ &= \frac{-10 \pm \sqrt{4 \cdot 43}}{-6} = \frac{-10 \pm 2\sqrt{43}}{-6} = \frac{5 \pm \sqrt{43}}{3}. \end{aligned}$$

And it should be immediately clear that the y-intercept is 6. □

A couple harder examples

Example 5. Find the standard equation of the parabola with vertex $(3, 5)$ and passing through $(4, 10)$.

Solution. We already know that the parabola looks like $y = a(x - 3)^2 + 5$. To find a, we know that when $x = 4$, $y = 6$. So

$$10 = a(4 - 3)^2 + 5,$$

which when we solve, we get $a = 5$. So our final answer is $y = 5(x - 3)^2 + 5$. □

Example 6. Find the standard equation of the parabola with zeros -7 and 4, and passing through the point $(-6, -9)$.

Solution. This problem lends itself to starting out in factored form. We haven't discussed this so far, but another way we can write a parabola is

$$y = a(x - x_1)(x - x_2),$$

where a is the same a we've been seeing, and x_1, x_2 are the zeros of the parabola. If you "foil" out this form, you'll see that the a is the coefficient of x^2, and it is pretty easy to see that plugging in x_1 or x_2 gives 0.

So for this problem, we have $y = a(x + 7)(x - 4)$. Using that the parabola passes through $(-6, -9)$, we see

$$-9 = a(-6 + 7)(-6 - 4) = a(1)(-10) = -10a \Rightarrow a = \frac{9}{10}.$$

Now we need to convert this to standard form:

$$\begin{aligned}
y &= \frac{9}{10}(x+7)(x-4) \\
&= \frac{9}{10}(x^2 + 3x - 28) \\
&= \frac{9}{10}\left(x^2 + 3x + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 - 28\right) \\
&= \frac{9}{10}\left(\left(x + \frac{3}{2}\right)^2 - \frac{9}{4} - 28\right) \\
&= \frac{9}{10}\left(x + \frac{3}{2}\right)^2 - \frac{9}{10}\frac{9}{4} - \frac{9}{10}(28) \\
&= \frac{9}{10}\left(x + \frac{3}{2}\right)^2 - \frac{1089}{40}
\end{aligned}$$

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