Inverse functions

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MA 158 Lesson 16

What is an inverse?

Definition 1. If f is a function, then the inverse of f is a function g such that $(g \circ f)(x) = x$ and $(f \circ g)(x) = x$. That is for every x, $f(g(x)) = g(f(x)) = x$.

Remark. Such a function g is unique denoted by f^{-1}.

Definition 2. A function f is said to be one-to-one (1-1) if $f(a) = f(b)$ implies $a = b$. That is, different inputs always correspond to different outputs.

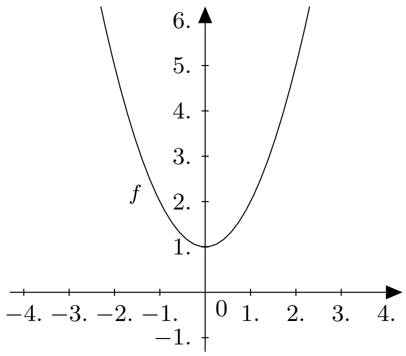
Remark. Graphically speaking, a function is one-to-one if it passes the horizontal line test. This is just what it sounds like and akin to the vertical line test. We imagine making an infinite horizontal line and travel up and down the graph. If that line hits at most one point of the graph at a time, then the function passes the horizontal line test.

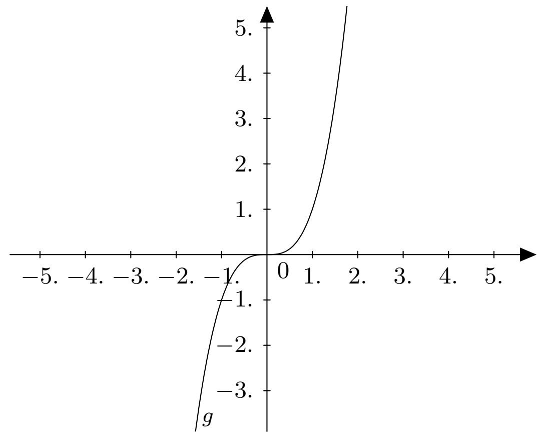
General idea

The inverse of a function f "undoes" what f did. Moreover, f is invertible if and only if it is one-to-one. In other words, f will have an inverse if and only if it passes the horizontal line test.

Example 1. Consider the function $f(x) = 3x + 7$. To get a y-value, we take an x, multiply it by 3, then add 7 to it. So to undo this, we would subtract 7 then divide by 3 to get back to our original x. Thus $f^{-1}(x) = \frac{x-7}{3}$.

Example 2. As a graphical example (below), f is not invertible, but g is. Note that we could restrict the domain of f to make it invertible. For example, if we only considered the interval $[0, \infty)$, then f would actually be invertible.





Finding the inverse

There are a couple of important things to notice about the inverse. As you may have noticed with Example 1, the roles of x and y are switched. Graphically, this means that f^{-1} will be a reflection of f about the line $y = x$. Further, since y and x are switched, the domain of f will be the range of f^{-1} and the range of f will be the domain of f^1.

Procedure. To actually find the inverse, we will follow a simple process. Given a function $y = f(x)$,

1. Switch x and y in the equation for $y = f(x)$.
2. Solve for y.

Example 3. Given $f(x) = 53x + 46$, find f^{-1} and the domain and range of f^{-1}.

Solution. $y = 53x + 46$, so we write

$$\begin{aligned} x &= 53y + 46 \\ x - 46 &= 53y \\ \frac{x-46}{53} &= y. \end{aligned}$$

So $f^{-1}(x) = \frac{x-46}{53}$. It should be clear that both the domain and range are $(-\infty, \infty)$. $\square$

Example 4. Given $g(x) = 90x^2 - 100,\ x \geq 0$, find g^{-1} and its domain and range.

Solution. First we should remark why the problem asserts that $x \geq 0$. Well, if we look at Example 2 again, it should be clear that quadratics don't generally pass the horizontal line test. So this domain restriction will allow us to find the inverse.

Now,

$$\begin{aligned} x &= 90y^2 - 100 \\ x + 100 &= 90y^2 \\ \frac{x+100}{90} &= y^2 \\ \sqrt{\frac{x+100}{90}} &= y. \qquad (*) \end{aligned}$$

Note that we only want the positive square root in $(*)$ since the domain of g was $x \geq 0$. So $g^{-1}(x) = \sqrt{\frac{x+100}{90}}$. Remembering that the domain of g is the range of g^{-1}, we can say that the range of g^{-1} is $[0, \infty)$. And the domain of g^{-1} is $[-100, \infty)$ since we need the inside of the square root to be nonnegative. □

Remark. Sometimes it may be easier to look at the original function to help us find the the domain and/or range of the inverse.

Example 5. Given $f(x) = \frac{x+24}{x+43}$, find f^{-1} and find its domain and range.

Solution.

$$\begin{aligned} x &= \frac{y+24}{y+43} \\ (y+43)x &= y + 24 \\ xy + 43x &= y + 24 \\ 43x - 24 &= y - xy \\ 43 - 24x &= y(1-x) \\ \frac{43-24x}{1-x} &= y = f^{-1}(x). \end{aligned}$$

Note that we could also write f^{-1} slightly differently:

$$f^{-1}(x) = \frac{43-24x}{1-x} = \frac{-(24x-43)}{-(x-1)} = \frac{24x-43}{x-1}.$$

The point here is that your final answer can look a couple of different ways, and they are equally correct. In Loncapa, it will not matter how you input your answer, but when it comes time for a test, you should be comfortable with factoring out negatives to verify that your answer matches an answer in a multiple choice question.

Now for the domain and range. We should be comfortable saying that the domain of f is $(-\infty, -43) \cup (-43, \infty)$. So this is the range of f^{-1}. And the domain of f^{-1} is $(-\infty, 1) \cup (1, \infty)$. We could have also taken advantage of looking for the horizontal asymptote of f to determine the range of f (and the domain of f^{-1}). □