Exponential functions

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MA 158 Lesson 17

Overview

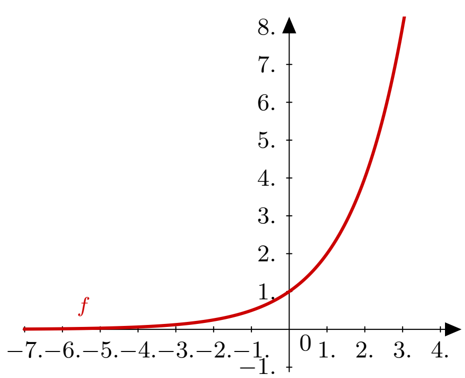
By now we should be comfortable with polynomials and rational functions. In both of these we have sums of powers of x. (Recall that a polynomial is something that looks like $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$.) Now we want to look at functions where x is in the exponent. Such a function is called an *exponential function*. Let us first consider the function $f(x) = 2^x$. To get a feel for what this will look like, we can plot some points.

x	$f(x)$
-3	$2^{-3} = 1/2^3 = 1/8$
-2	$2^{-2} = 1/2^2 = 1/4$
-1	$2^{-1} = 1/2^1 = 1/2$
0	$2^0 = 1$
1	$2^1 = 2$
2	$2^2 = 4$
3	$2^3 = 8$

We could keep going, but it should be clear that as $x \to \infty$, i.e., x gets very large, $f(x) \to \infty$ as well. What about as $x \to -\infty$? Well,

$$f(x) = 2^{\text{very negative number}} = \frac{1}{2^{\text{very large positive number}}},$$

so $f(x) \to 0$. Now we should be able to see that the graph of f looks like the one below.



Properties of the exponential

There was nothing special about 2^x, and more generally, we can consider an exponential function $f(x) = b^x$, where b is a real number. The number b is called the *base* of the exponential. And we have the following properties:

- $b^0 = 1$
- $b^x b^y = b^{x+y}$
- $(b^x)^y = b^{xy}$
- $b^{-x} = \frac{1}{b^x}$
- $b^{1/x} = \sqrt[x]{b}$
- If $b^x = b^y$, then $x = y$. Recall that this means that the exponential function is 1-1, so it has an inverse, which will be covered next time.
- Domain is $(-\infty, \infty)$
- Range is $(0, \infty)$ since we have a horizontal asymptote of $y = 0$, and the graph never crosses the horizontal asymptote.
- If $0 < b < 1$ then the exponential is decreasing. If $b > 1$ then the exponential is increasing.

The natural exponential e^x

There are a few important constants found in the world of mathematics. One of them is Euler's number e. There is a "fun" mnemonic for memorizing the first 15 decimal places of e: We need to know that it starts with 2 and a few facts about Andrew Jackson. So Jackson was the 7th president of the US, so we have 2.7. He was elected in 1828 for the second time, so we have 2.718281828, and if we are looking at a \$20 bill and make a square box around his face and draw a diagonal, the angles will be 45-90-45. Now we have that

$$e \approx 2.718281828459045.$$

The important thing to take from this is that e is a *number*. Further, e has all the same properties as the general exponential we have just discussed. In addition it has several more special properties. Some of these will be uncovered during this course, others when you take calculus. For the impatiently curious, you can find out quite a bit just checking out the Wikipedia page for e.

We should be comfortable manipulating equations with e.

Example 1. Simplify the following.

(a) $e^9 e^{18}$

(b) $e^{-x}e^{-4}$

(c) $(e^x)^2$

Solution. (a) $e^9e^{18} = e^{9+18} = e^{27}$. Note that we should leave this as e^{27}. Plugging this into your calculator will give you a decimal *approximation*, and will no longer be an exact answer.

(b) $e^{-x}e^{-4} = e^{-x-4}$. We could also write this as $e^{-(x+4)} = 1/e^{x+4}$. Any of these three answers would be considered equally correct, but you should be comfortable converting between them.

(c) $(e^x)^2 = e^{2x}$. □

Example 2. Simplify the following.

(a) $(e^{7x} + e^{-7x})(e^{7x} - e^{-7x})$

(b) $(e^{18x} - e^{-13x})^2$

Solution. (a)

$$\begin{aligned}(e^{7x} + e^{-7x})(e^{7x} - e^{-7x}) &= e^{7x}e^{7x} - e^{7x}e^{-7x} + e^{-7x}e^{7x} - e^{-7x}e^{-7x} \\ &= e^{14x} - 2e^{-14x}\end{aligned}$$

(b) $(e^{18x} - e^{-13x})^2 = e^{18x}e^{18x} - 2e^{18x}e^{-13x} + e^{-13x}e^{-13x} = e^{36x} - 2e^{5x} + e^{-26x}$. □

Applications of the exponential

Simple compound interest

The most ubiquitous application of exponential functions are the interest formulas. If you invest money in some sort of account, interest is compounded at some interval. The mathematical formula to figure out how much you will earn is given by

$$A = P(1 + \frac{r}{n})^{nt},$$

where P is the principal, or initial amount, t is the number of years, n is the number of times interest is compounded per year, r is the rate (as a decimal), and A is the final amount after t years.

Example 3. Suppose you invest \$20,000 at 10% compounded weekly for 15 years.

(a) How much will you have at the end of 15 years?

(b) How much interest will you have earned?

Solution. Here, $P = 20,000,\ r = 0.1,\ n = 52$ and $t = 15$. So for (a),

$$A = 20000\left(1 + \frac{0.1}{52}\right)^{52\cdot 15} \approx \$89,504.76.$$

To find the interest, we simply compute $A - P = 89504.76 - 20000 = \$69,504.76$. □

Continuous compounding

It is a natural question to ask: what happens if the number of times we compound interest per year gets really big? As $n \to \infty$, the simple compound interest formula becomes

$$A = Pe^{rt},$$

where each symbol has the same meaning as before. When it comes time for a quiz or exam, you will be given both the simple compound interest and continuous compounded interest formulas.

Example 4. A certain off-shore banking account earns a yearly interest rate of 13% compounded continuously. How much money should be invested so that $141,101,311 will be in the account after 6 years?

Solution. Here $r = 0.13$, $A = 141,101,311$ and $t = 6$. Since $A = Pe^{rt}$, we can solve for P and we have $P = A/e^{rt}$. So

$$P = \frac{141101311}{e^{0.13 \cdot 6}} \approx \$64,681,689.17.$$

□

Example 5. Americium-241 is a ubiquitous isotope of Am, and is probably found in your household smoke detector. A sample of Am-241 decays according to the model

$$N(t) = 47.31e^{-.00160376t},$$

where $N(t)$ is the amount remaining in grams and t is time in years.

(a) What is the initial mass of the sample?

(b) What mass will remain after remain after 484 days?

(c) What mass will remain after 200 years?

(d) What percentage of the mass will remain after 432.2 years?

(a) The initial amount is 47.31 grams

(b) 484 days is $\frac{484}{365}$ years, so

$$N\left(\frac{484}{365}\right) = 47.31e^{-.00160376 \cdot \frac{484}{365}} \approx 47.21 \text{ grams}$$

(c) $N(200) \approx 34.33$ grams

(d) $N(432.2) \approx 23.655$. To find the percent remaining, we compute

$$N(432.2)/N(0) \approx 0.5.$$

So there is 50% remaining.