Applications of log

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MA 158 Lesson 21

Probably more so than anything we have done up to now, exponential and log functions give rise to many useful applications. To ease into the lesson, let's start with something that requires no new information. By this point, we should be fairly comfortable manipulating exponential and logarithmic equations.

Example 1. Solve the following equation for t.

$$a = b(8 - e^{ct})$$

Solution. We just want to isolate t.

$$\begin{aligned}
a &= b(8 - e^{ct}) \\
\frac{a}{b} &= 8 - e^{ct} \\
e^{ct} &= 8 - \frac{a}{b} \\
ct &= \ln\left(8 - \frac{a}{b}\right) \\
t &= \frac{1}{c}\ln\left(8 - \frac{a}{b}\right).
\end{aligned}$$

Note that in order to have a solution for t, we require that $b \neq 0$ (which we need from the statement of the problem), we also require $c \neq 0$ and because of the domain restriction from log, we need $a/b < 8$. □

One important type of example is that of $\frac{1}{2}$-life. In a previous lesson, we used the $A = Pe^{rt}$ equation to model continuous compounding. In fact, we can model any exponential growth or decay using this exact expression. Typically the letters get changed a bit, but the essence is still there. So we will be using

$$A(t) = A_0 e^{kt}$$

to model exponential growth (decay). In this equation $A(t)$ represents the amount at time t, A_0 is the initial amount, k is the rate of growth (decay), and t is time. In the interest problems we required that t be in years, but for other applications, we don't make such an assertion. So t can be any unit of time, and k will be some constant rate which depends on the unit of time we use.

In case you don't know, $\frac{1}{2}$-life refers to the amount of time it takes for some (generally radioactive) substance to decay to half of its original mass. For example Carbon-14 has a $\frac{1}{2}$-life of about 5730 years. When scientists try to determine how old certain things are, they look at the amount of ^{14}C is present and estimate the age with their knowledge of how quickly ^{14}C decays.

Example 2. You asked for gold for Christmas and foolishly did not specify which isotope. As a result, your friend gives you a 1kg sample of ^{196}Au, which has a half-life of 148.39 hours.

(a) Find a function $A(t)$ for the amount of the isotope, A in grams, which remains after time t in hours.

(b) Determine the time t in hours for 93% of the material to decay.

Solution. (a) As always, it is important to pay attention to units. We want A to be in grams, but we are told that we start with 1 kg. So then $A_0 = 1000$ g. Now we know that the function is of the form

$$A(t) = 1000e^{kt},$$

but we don't know what k is. To find k, we know that whenever $t = 148.39$, then we have half of what we started with. So then

$$\begin{aligned} 500 &= 1000e^{148.39k} \\ \frac{1}{2} &= e^{148.39k} \\ \ln\left(\frac{1}{2}\right) &= 148.39k \\ k &= \frac{1}{148.39}\ln\left(\frac{1}{2}\right) \approx -0.00467 \end{aligned}$$

So then our function is

$$A(t) = 1000e^{-0.00467t}.$$

(b) If 93% of the material has *decayed*, that means that there is only 7% remaining. One way to set up this kind of problem is to think of our initial amount as 100%. Then this looks like

$$7 = 100e^{-0.00467t}.$$

And now we want to solve for t.

$$\begin{aligned}
7 &= 100e^{-0.00467t} \\
\frac{7}{100} &= e^{-0.00467t} \\
\ln\left(\frac{7}{100}\right) &= -0.00467t \\
t &= \frac{1}{-0.00467}\ln\left(\frac{7}{100}\right) \approx 569.298 \text{ hours}
\end{aligned}$$ □

Example 3. The radioactive isotope ^{93}Sr has a half-life of 7.5 minutes. Find how long it will take for a sample to decay so that 63% of its original mass remains.

Solution. In a situation like this, we must use the strategy from Example 2(b) as we are not given an initial mass. Just for variety, another way to think about it than in terms of percents is that when $t = 7.5$, then $A(t) = \frac{1}{2}A_0$. So then we may write

$$\frac{1}{2}A_0 = A_0e^{7.5t},$$

And when we divide both sides by A_0, we will get A_0/A_0 which is 1. Now

$$\begin{aligned}
\frac{1}{2}A_0 &= A_0e^{7.5t} \\
\frac{1}{2} &= e^{7.5t} \\
\ln\left(\frac{1}{1}\right) &= 7.5t \\
t &= \frac{1}{7.5}\ln\left(\frac{1}{1}\right) \approx -0.0924.
\end{aligned}$$

So then the equation for the decay of the isotope is

$$A(t) = A_0e^{-0.0924t}.$$

Now to find at what time 63% is *remaining*, it probably makes the most sense to view masses as percentages again. So we have

$$\begin{aligned}
63 &= 100e^{-0.0924t} \\
.63 &= e^{-0.0924t} \\
\ln(.63) &= -0.0924t \\
t &= \frac{1}{-0.0924}\ln(.63) \approx 5.00 \text{ minutes.}
\end{aligned}$$ □

Population growth, especially that of bacteria and spread of disease are well-modeled by exponential growth functions. This next example is nice because we need to be able to interpret real-world information to translate it to the equations we are familiar with.

Example 4. In ideal conditions, the spread of disease follows the Law of Uninhibited Growth

$$N(t) = N_0 e^{kt}.$$

During the 2014 West Africa Ebola outbreak, there were an estimated 800 reported cases of Ebola on July 1, and 1500 on August 1 (31 days later). Let t be the time elapsed in days since the first observation.

(a) Find the growth constant k. Round your answer to four decimal places.

(b) Find a function which gives the number of reported cases $N(t)$ after t days.

(c) What is the doubling time for the number of reported cases?

Solution. (a) So here $N_0 = N(0) = 800$ and $N(31) = 1500$. So we already know that the function looks like

$$N(t) = 800e^{kt}.$$

We just need to determine k. You should notice that as soon as we determine k, we immediately have a solution for part (b). Now using the second piece of data, we know

$$\begin{aligned} 1500 &= 800e^{31k} \\ \frac{15}{8} &= e^{31k} \\ \ln\left(\frac{15}{8}\right) &= 31k \\ k &= \frac{1}{31}\ln\left(\frac{15}{8}\right) \approx .02. \end{aligned}$$

(b) Here we just need to put it together, so $N(t) = 800e^{.02t}$.

(c) To figure out the doubling time is quite similar to figuring out the half-life. Whenever we have an initial amount N_0, we are looking for the time t for which $N(t) = 2N_0$. So this looks like

$$2N_0 = N_0 e^{kt}.$$

In this particular example, we know what N_0 is, so we can just say we want to solve $N(t) = 1600$ for t. But this shows that we don't actually need to know the initial amount to figure out the doubling time.

Back to the problem at hand:

$$\begin{aligned} 1600 &= 800e^{.02t} \\ 2 &= e^{.02t} \\ \ln 2 &= .02t \\ t &= \frac{\ln 2}{.02} \approx 34.66 \text{ days.} \end{aligned}$$

□

For our final example, we turn to logistic growth, which has several applications, one of which is modeling population growth of cities.

Example 5. The population of a certain town is modeled by

$$P(t) = \frac{670}{1 + 6.55e^{-0.51t}}$$

where $P(t)$ is the population t years after 2010. Such a model is called logistic growth (decay if we e had a positive exponent).

(a) What is the population in 2010?

(b) Find the population in 2012.

(c) When will the population reach 310?

Solution. (a) This is just asking for $P(0)$.

$$P(0) = \frac{670}{1 + 6.55e^0} = \frac{670}{7.55} \approx 89.$$

We round to the nearest whole number under the assumption that we can't have a fraction of a person.

(b) This is just asking for $P(2)$.

$$P(2) = \frac{670}{1 + 6.55e^{-0.51 \cdot 2}} \approx 199.$$

(c) Here we want to solve for t in the expression $P(t) = 310$.

$$\begin{aligned}
310 &= \frac{670}{1 + 6.55e^{-0.51t}} \\
1 + 6.55e^{-0.51t} &= \frac{670}{310} \\
6.55e^{-0.51t} &= \frac{670}{310} - 1 \\
e^{-0.51t} &= \frac{1}{6.55}\left(\frac{670}{310} - 1\right) \\
-0.51t &= \frac{1}{6.55}\left(\frac{670}{310} - 1\right) \\
t &= \frac{-1}{0.51}\left(\frac{1}{6.55}\left(\frac{670}{310} - 1\right)\right) \approx 3.39.
\end{aligned}$$

We have to interpret this as years since 2010, so our final answer is 2013. □