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MA 158
Lesson 6

Function arithmetic

Recall that we define $+, -, \cdot, /$ on functions by performing these operations on the outputs. So we have

- $(f+g)(x) = f(x) + g(x)$
- $(f-g)(x) = f(x) - g(x)$
- $(fg)(x) = f(x)g(x)$
- $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, provided that $g(x) \neq 0$.

As an example, let $f(x) = x^2 + 1$ and $g(x) = 2x - 4$. Then

- $(f+g)(x) = (x^2+1) + (2x-4) = x^2 + 2x - 3$
- $(f-g)(x) = (x^2+1) - (2x-4) = x^2 - 2x + 5$
- $(fg)(x) = (x^2+1)(2x-4) = 2x^3 - 4x^2 + 2x - 4$, note that it is not necessary to multiply this out. You could leave it in factored form.
- $\left(\frac{f}{g}\right)(x) = \frac{x^2+1}{2x-4}$

We could also ask for something like $(f+g)(3)$ or $(fg)(3)$, etc. So, we could say $(f+g)(3) = f(3) + g(3)$ and find $f(3)$ and $g(3)$ separately. This is beneficial when we don't actually care what $(f+g)(x)$ is in general and only care about a specific value.

Composition of functions

Composition of functions is another way of saying that we perform a function on a function. There is an "inside" function and an "outside" function. The notation we use is $f \circ g$, and we read this as "f composed of g" or "f composed with g." What this notation means in practice is

$$(f \circ g)(x) = f(g(x)).$$

Here, first we apply the function g to our input x, then we apply the function f to $g(x)$; so the output of g is the input for f. Visually speaking, g is the "inside" function, and f is the outside function.

Example 1. $f(x) = 2x + 3$ and $g(x) = x + 4$. Then

(a) $(f \circ g)(1) = f(g(1)) = f(1 + 4) = f(5) = 2 \cdot 5 + 3 = 13$

(b) $(f \circ g)(x) = f(g(x)) = f(x + 4) = 2(x + 4) + 3 = 2x + 11$

(c) $(g \circ f)(x) = g(f(x)) = g(2x + 3) = (2x + 3) + 4 = 2x + 7$.

Notice. Function composition is not commutative, i.e., $(f \circ g)(x) \neq (g \circ f)(x)$. This is a very important point; it *does* matter the order in which you perform functions.

Example 2. Use the table table to complete the following:

x	1	2	3	4	5	6
f(x)	7	5	2	4	6	3
g(x)	9	4	5	3	2	6

(a) $2f(3) + 5g(3)$

(b) $2(f - g)(5)$

(c) $(f \circ g)(3)$

(d) $(g \circ f)(3)$

(You should get (a) 29, (b) 8, (c) 6, (d) 4.)

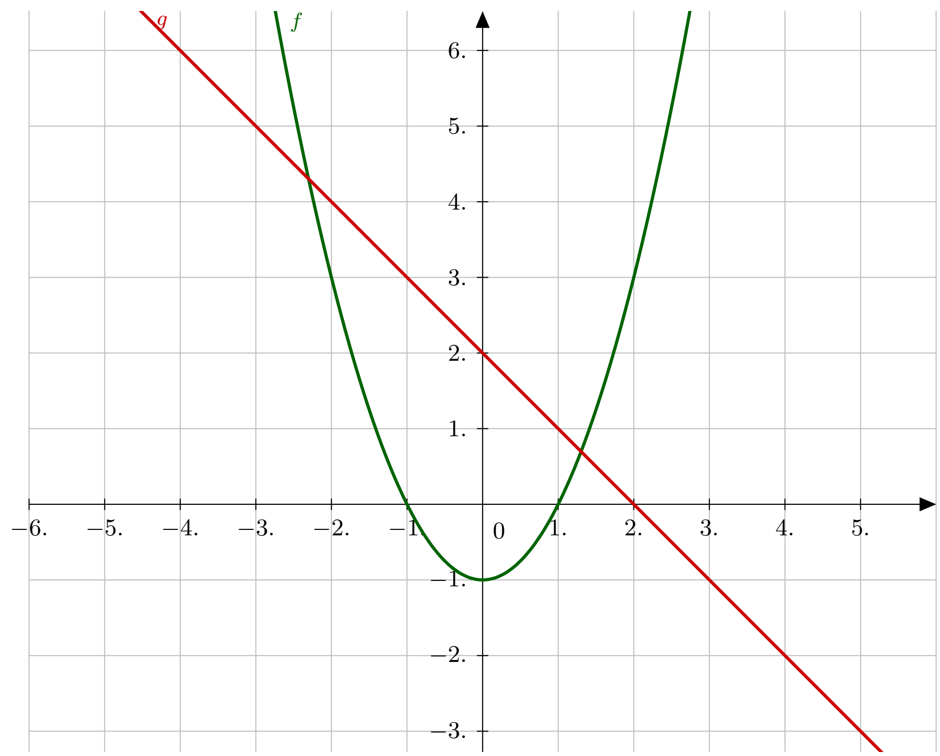
Example 3. Consider the following graph and find

(a) $(f - g)(2)$

(b) $(fg)(-1)$

(c) $(f \circ g)(-3)$

(d) $(g \circ f)(-1)$.



We didn't get the chance to go through this one in one class, so we'll go through this in a little more detail. (a), (b) and (d) should be fairly straightforward using the tools from the previous examples and just looking at the graph. Part (c) is where many students may have trouble and is similar to the homework. We'll do that one now.

Recall that $(f \circ g)(-3) = f(g(-3))$. It's easy to see that $(-3) = 5$. But $f(5)$ is off the chart. How can we figure out $f(5)$ then? The best way is to determine what the equation for f is.

In general, whenever we have a parabola, p with two x-intercepts, say x_1, x_2, then the equation for p is given by

$$p(x) = a(x - x_1)(x - x_2),$$

where a is some scaling factor. We can determine the value of a by plugging in 0 for x. Then we see that

$$p(0) = a(0 - x_1)(0 - x_2) = a(-x_1)(-x_2) = ax_1x_2,$$

and we know the values for $p(0), x_1, x_2$, so $a = p(0)/(x_1x_2)$.

In this particular case, $p(0) = -1$, and $x_1 = 1, x_2 = -1$, so $a = \frac{-1}{(1)(-1)} = 1$. So $f(x) = (x-1)(x+1)$. So $f(5) = (5-1)(5+1) = 4 \cdot 5$.

Domain of compositions

Whenever we add, subtract, multiply or divide two functions, notice that for x to be in the domain of $f\,(+, -, \cdot, /)\,g$, we need x to be in the domain of f and the domain of g.

But when we want x to be in the domain of $f \circ g = f(g(x))$, then we need x to be in the domain g and $g(x)$ to be in the domain of f. Notice that we don't actually need x to be in the domain of f.

Example 4. $f(x) = \frac{x+3}{x-2}$ and $g(x) = \frac{2x+1}{3x-4}$. Then

$$(f \circ g)(x) = \frac{\frac{2x+1}{3x-4} + 3}{\frac{2x+1}{3x-4} - 2}. \qquad (*)$$

From here it is clear that we need to have $3x - 4 \neq 0$ and $\frac{2x+1}{3x-4} - 2 \neq 0$. In the first case we need $x \neq \frac{4}{3}$. In the second case

$$\begin{aligned} &\frac{2x+1}{3x-4} - 2 \neq 0 \\ &\Leftrightarrow \frac{2x+1}{3x-4} \neq 2 \\ &\Leftrightarrow 2x + 1 \neq 2(3x - 4) \\ &\Leftrightarrow 2x + 1 \neq 6x - 8 \\ &\Leftrightarrow 4x \neq 9 \\ &\Leftrightarrow x \neq \frac{9}{4}. \end{aligned}$$

So then the domain of $f \circ g$ is all real numbers except $\frac{4}{3}$ and $\frac{9}{4}$. In interval notation, that is $(-\infty, 4/3) \cup (4/3, 9/4) \cup (9/4, \infty)$. Similarly, for $g \circ f$, you should get that the domain is $(-\infty, 2) \cup (2, 17) \cup (17, \infty)$.

Remark. We can apply a previous lesson to simplify our expression for $(f \circ g)(x)$, which we should do if the instructions on a homework, quiz or exam say to do so. Recall how to do this. We can multiply our expression in $(*)$ by $\frac{3x-4}{3x-4}$. So

$$\begin{aligned} f(g(x)) = \frac{\frac{2x+1}{3x-4} + 3}{\frac{2x+1}{3x-4} - 2} \cdot \frac{3x-4}{3x-4} &= \frac{(2x+1) + 3(3x-4)}{(2x+1) - 2(3x-4)} \\ &= \frac{(2x+1) + 9x - 12}{(2x+1) - 6x + 8} = \frac{11x - 11}{-4x + 9}. \end{aligned}$$

Example 5. $f(x) = \sqrt{12 - x}$ and $g(x) = \sqrt{x+1}$. Then

$$(f \circ g)(x) = f(g(x)) = f(\sqrt{x+1}) = \sqrt{12 - \sqrt{x+1}}.$$

Then the domain of g is all $x \geq -1$. For $f(g(x))$, we need $12 - \sqrt{x+1} \geq 0$. That is,

$$\begin{aligned} \sqrt{x+1} &\leq 12 \\ x + 1 &\leq 144 \\ x &\leq 143. \end{aligned}$$

Putting this together with $x \geq -1$, we get $-1 \leq x \leq 143$, or in interval notation, $[-1, 143]$. Similarly, $(g \circ f)(x) = g(f(x)) = g(\sqrt{12-x}) = \sqrt{\sqrt{12-x}+1}$. Here, we need $x \leq 12$ to be sure that x is in the domain of f. For g, we need $\sqrt{12-x}+1 \geq 0$. That is, $\sqrt{12-x} \geq -1$. Notice that there is no problem here since we already know $\sqrt{12-x} \geq 0$. So as it turns out, the only restriction is $x \leq 12$, or in interval notation, $(-\infty, 12]$.

Remark. You should leave these nasty-looking square root expressions as they are. There is no reason and no good way to make them look prettier.

Example 6. $f(x) = 10x^2 + 7$ and $g(x) = \frac{1}{x}$. Then

$$(f \circ g)(x) = f(g(x)) = f\left(\frac{1}{3x}\right) = \frac{10}{9x^2} + 7.$$

Then the domain is $(-\infty, 0) \cup (0, \infty)$. For $g \circ f$,

$$(g \circ f)(x) = g(f(x)) = g(10x^2 + 7) = \frac{1}{3(10x^2+7)}.$$

Then it's easy to verify that the denominator is never equal to 0, so the domain here is $(-\infty, \infty)$.