The difference quotient

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MA 158 Lesson 8

Warm-up. Given

$$f(x) = \frac{x+3}{x^2-1} \quad \text{and} \quad g(x) = \frac{1}{x},$$

compute $(g \circ f)(x)$ and find its domain.

Solution.

$$(g \circ f)(x) = g(f(x)) = g\left(\frac{x+3}{x^2-1}\right) = \frac{1}{\frac{x+3}{x^2-1}} = \frac{x^2-1}{x+3} \tag{1}$$

To find the domain of $g(f(x))$ we need to determine two things:

1. What values x can we plug into f, and
2. What values $f(x)$ can we plug into g?

To deal with the first question, let's look at $f(x)$. Since this is a rational function, we need to check when is the denominator zero. Setting $x^2 - 1 = 0$, we see $x^2 = 1$, so square rooting both sides, gives $x = \pm 1$. So we can't have $x = \pm 1$.

To deal with the second question, we only want to talk about x values, so we look at the expression in (1). Now the question becomes what values of x can we plug into *this* expression (since we have already plugged in $f(x)$). Looking after the final equal sign, it should be clear that we can't have $x = -3$ as that gives a 0 in the denominator.

Combining our results from 1. and 2., we can't have $x = -3, -1, 1$. So the domain of $g \circ f$ is $(-\infty, -3) \cup (-3, -1) \cup (-1, 1) \cup (1, \infty)$. $\square$

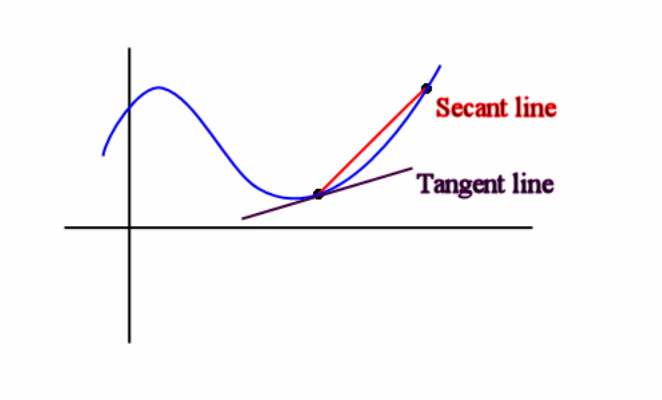
The motivation

The motivation behind today's lesson is calculus. We already know how to find the slope of a line: $\frac{\Delta y}{\Delta x}$. It is a natural question to ask about the slope of a curve. But this is a bit more complicated since the slope of a curve is always changing, unlike a line, which has constant slope. Before we proceed we have a couple of new words to know.

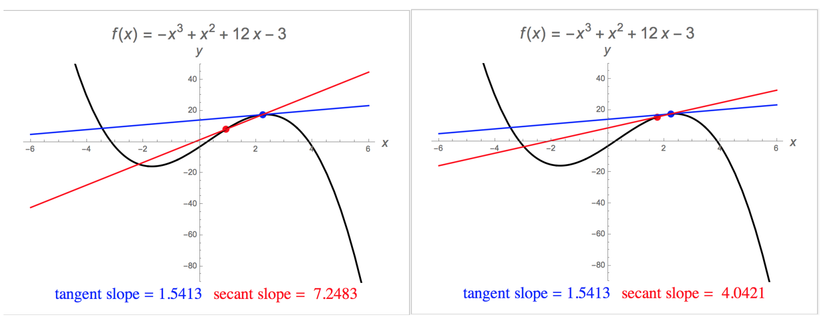
Definition 1. A <u>tangent line</u> to a graph is one which touches the graph at a single point.

Definition 2. A <u>secant line</u> to a graph is one which touches the graph at two points.

The graph below helps to illustrate what exactly this means. Another way to think about the secant line is to pick any two points on the graph and draw a line between them. This will be a secant line.



So to find the slope at a particular point, we we start with the slope of a secant line, and move the one of the points closer to the other to keep getting closer to the tangent line's slope. Here is a more concrete example below.



To the point

In this class we are only concerned with the difference quotient. For any function f, if we want to find the slope of the secant line between x_1 and x_2, we use the equation for slope we've seen since middle school:

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1}. \tag{2}$$

If we write $h = x_2 - x_1$, then we can say $x_2 = x_1 + h$. Now we can rewrite (2) as

$$\frac{f(x_1 + h) - f(x_1)}{h} = \frac{f(x + h) - f(x)}{h}, \tag{3}$$

where the right hand side is just relabeling x_1 as x. Alternatively, you may see the difference quotient written as

$$\frac{f(x) - f(a)}{x - a},$$

but we're generally going to stick with the format in (3). If you have had any exposure to calculus, you may recognize the difference quotient in relation to the derivative (the slope of the tangent line at a particular point). If we were to take the limit as $h \to 0$, then this would be the derivative. If this last part means nothing to you, don't worry about that.

All we need to be able to do is find the difference quotient for different functions and simplify.

Remark. We are not done simplifying the difference quotient until the original h in the denominator has been cancelled out somehow. When we are working in the numerator, all terms that do not have an h should get cancelled out. If this does not happen, you should check your work.

Example 1. Find the average rate of change of $f(x) = x^3$ on the interval $[19, 45]$.

Solution. Whenever you hear "average rate of change" you should be thinking "slope of the secant line." So we just use our usual slope formula:

$$\frac{f(45) - f(19)}{45 - 19} = \frac{45^3 - 19^3}{45 - 19} = 3241. \qquad \square$$

Example 2. Find the average rate of change of $f(x) = x^3$ on the interval $[x, x + h]$.

Solution. Here we do the same thing, but instead of numbers, we use x and $x + h$ for our "x" values.

$$\begin{aligned}
\frac{(x+h)^3 - x^3}{(x+h) - x} &= \frac{(x+h)(x+h)(x+h) - x^3}{h} \\
&= \frac{(x^2 + 2xh + h^2)(x+h) - x^3}{h} \\
&= \frac{x^3 + 3xh^2 + 3x^2h + h^3 - x^3}{h} \\
&= \frac{3xh^2 + 3x^2h + h^3}{h} = \frac{3xh^2}{h} + \frac{3x^2h}{h} + \frac{h^3}{h} \\
&= 3x^2 + 3xh + h^2
\end{aligned} \qquad \square$$

Example 3. Find and simplify the difference quotient for $f(x) = -5x^2 - 8x - 1$.

Solution.

$$\begin{aligned}
&\frac{-5(x+h)^2 - 8(x+h) - 1 - [-5x^2 - 8x - 1]}{h} \\
&= \frac{-5x^2 - 10xh - 5h^2 - 8x - 8h - 1 + 5x^2 + 8x + 1}{h} \\
&= \frac{-10xh - 5h^2 - 8h}{h} \\
&= -10x - 5h - 8
\end{aligned} \qquad \square$$

Example 4. Find and simplify the difference quotient for $f(x) = \sqrt{4x}$.

Solution. In this problem it should become clear why we were concerned with rationalizing radial expressions. We will use the strategy of multiplying by the conjugate just as in a previous lesson.

$$\begin{aligned}
&\frac{\sqrt{4(x+h)} - \sqrt{4x}}{h} \cdot \frac{\sqrt{4(x+h)} + \sqrt{4x}}{\sqrt{4(x+h)} + \sqrt{4x}} \\
&\qquad = \frac{4(x+h) - 4x}{h(\sqrt{4(x+h)} + \sqrt{4x})} \\
&\qquad = \frac{4x + 4h - 4x}{h(\sqrt{4(x+h)} + \sqrt{4x})} \\
&\qquad = \frac{4h}{h(\sqrt{4(x+h)} + \sqrt{4x})} \\
&\qquad = \frac{4}{\sqrt{4(x+h)} + \sqrt{4x}}
\end{aligned}$$

□

Remark. An h does show up in the denominator here, but it is *not* the h that was original set up for the difference quotient. A good test is to plug in 0 for h. If you can do this, then your answer is completely simplified.

Summary

With these four examples you should be able to complete the homework and any difference quotient problem that you come across in this course. A couple things to keep in mind:

1. For polynomial functions, you will need to multiply out all terms like $(x+h)^3$ or $(x+h)^2$, etc. to be able to cancel out the terms you need to cancel out.

2. For square root functions, remember to multiply the top and bottom by the conjugate.

As a final note, if you are curious about how to quickly multiply out things like $(x+y)^n$, look up the binomial theorem on Wikipedia.