Overview

In this lesson we add the quotient rule and the remaining trig functions to our list of functions that we know how to differentiate.

Lesson

The quotient rule

As the name suggests, we consider a function of the form $y = \frac{u(x)}{v(x)}$.

If $y = u(x)v(x)$, then

$$\frac{dy}{dx} = \frac{u'(x)v(x) - u(x)v'(x)}{(v(x))^2},$$

or more succinctly,

$$\left(\frac{u}{v}\right)' = \frac{uv' - u'v}{v^2}.$$

Remark. Unlike the product rule, what we call u and what we call v matters, and the order in which we take the derivatives matters because of the minus sign in the numerator. Many people remember this rule using the mnemonic "low d high minus high d low all over low squared."

Other trig derivatives

The derivative rules of the remaining trig functions are much simpler than the product and quotient rules. We list them here:

1. $\frac{d}{dx} \sec x = \sec x \tan x$
2. $\frac{d}{dx} \csc x = -\csc x \cot x$
3. $\frac{d}{dx} \tan x = \sec^2 x$
4. $\frac{d}{dx} \cot x = -\csc^2 x$

Example 1. Find $\frac{dy}{dx}$, where

$$y = \frac{x^3}{x^4 + 4x + 9}.$$

Solution. Here $u = x^3$ and $v = x^4 + 4x + 9$, so that $u' = 3x^2$ and $v' = 4x^3 + 4$. Then

$$\begin{aligned}\frac{dy}{dx} &= \frac{3x^2\left(x^4 + 4x + 9\right) - x^3\left(4x^3 + 4\right)}{\left(x^4 + 4x + 9\right)^2} \\ &= \frac{3x^6 + 12x^3 + 27x^2 - 4x^6 - 4x^3}{(x^4 + 4x + 9)^2} \\ &= \frac{-x^6 + 8x^3 + 27x^2}{(x^4 + 4x + 9)^2}.\end{aligned}$$

□

Remark. While it is nice to simplify the numerator of the derivative, it is generally unnecessary to expand the denominator. (Nor does that make the fraction look any simpler.)

Example 2. Find $f'(\pi)$ where

$$f(x) = \frac{5\sin x - 5\cos x}{\sin x + 4\cos x}.$$

Solution. Here we have $u = 5\sin x - 5\cos x$ and $v = \sin x + 4\cos x$ so that $u' = 5\cos x + 5\sin x$ and $v' = \cos x - 4\sin x$. Then

$$f'(x) = \frac{(5\cos x + 5\sin x)(\sin x + 4\cos x) - (5\sin x - 5\cos x)(\cos x - 4\sin x)}{(\sin x + 4\cos x)^2}$$

and

$$\begin{aligned} f'(\pi) &= \frac{(5\cos\pi + 5\sin\pi)(\sin\pi + 4\cos\pi) - (5\sin\pi - 5\cos\pi)(\cos\pi - 4\sin\pi)}{(\sin\pi + 4\cos\pi)^2} \\ &= \frac{(5(-1) + 5(0))(0 + 4(-1)) - (5(0) - 5(-1))((-1) - 4(0))}{((0) + 4(-1))^2} \\ &= \frac{(-5)(-4) - (5)(-1)}{(-4)^2} \\ &= \frac{25}{16} \end{aligned}$$

□

Remark. It was far simpler to plug in $x = \pi$ and then to simplify the fraction than to try to simplify the general derivative. When we get expressions this complicated, it's usually just best to leave the answer as is unless we're evaluating the derivative at a particular point as in this example.

Example 3. Find $\frac{dy}{dx}$, where

$$y = \frac{8(a^2 + x^2)}{x^2 - a^2}$$

and a is a constant.

Solution. Here $u = 8(a^2 + x^2) = 8a^2 + 8x^2$ and $v = x^2 - a^2$ so that $u' = 16x$ and $v' = 2x$. Then using the quotient rule gives

$$\begin{aligned} \frac{dy}{dx} &= \frac{16x\left(x^2 - a^2\right) - 8\left(\left(a^2 + x^2\right)(2x)\right)}{\left(x^2 - a^2\right)^2} \\ &= \frac{16x^3 - 16a^2x - 16a^2x - 16x^3}{\left(x^2 - a^2\right)^2} \\ &= \frac{-32a^2x}{\left(x^2 - a^2\right)^2}. \end{aligned}$$

□

Example 4. Find the derivative of $y = \tan x \csc x$.

Solution. This is just an application of the product rule and the new derivative rules we learned. Let's make $u = \tan x$ and $v = \csc x$ so that $u' = \sec^2 x$ and $v' = -\csc x \cot x$. Then

$$\begin{aligned} y' &= \tan x(-\csc x \cot x) + \sec^2 x \csc x \\ &= -\csc x + \sec^2 x \csc x. \end{aligned}$$

In the last line here we used the fact that $\cot x = \frac{1}{\tan x}$. □

Example 5. Find the derivative of $y = \sec x \tan x$.

Solution. We need to be careful not to get things backward. The derivative of $\sec x$ is $\sec x \tan x$, but the derivative of $\sec x \tan x$ is not $\sec x$. Again we use the product rule. We'll make $u = \sec x$ and $v = \tan x$. Then $u' = \sec x \tan x$ and $v' = \sec^2 x$, and

$$\begin{aligned}\frac{dy}{dx} &= \sec x(\sec^2 x) + \sec x \tan x(\tan x)\\ &= \sec^3 x + \sec x \tan^2 x.\end{aligned}$$

□

Example 6. Find $f'\left(\frac{\pi}{4}\right)$, where

$$f(x) = \frac{7\cot x}{9 + 3\cos x}.$$

Solution. This is clearly a quotient rule problem. Here $u = 7\cot x$ and $v = 9 + 3\cos x$ so that $u' = -7\csc^2 x$ and $v' = -3\sin x$. Then

$$\begin{aligned}f'(x) &= \frac{-7\csc^2 x(9 + 3\cos x) - 7\cot x(-3\sin x)}{(9 + 3\cos x)^2}\\ &= \frac{-7\csc^2 x(9 + 3\cos x) + 21\cot x \sin x}{(9 + 3\cos x)^2}.\end{aligned}$$

□

Example 7. Find an equation of the tangent line to the graph of $y = 10x^8 \csc x$ at $x = \frac{\pi}{3}$

Solution. The derivative is a quick application of the product rule. We'll make $u = 10x^8$ and $v = \csc x$. Then $u' = 80x^7$ and $v' = -\csc x \cot x$, and

$$\frac{dy}{dx} = 10x^8(-\csc x \cot x) + 80x^7 \csc x.$$

Then

$$\begin{aligned}\left.\frac{dy}{dx}\right|_{x=\pi/3} &= 10\left(\frac{\pi}{3}\right)^8\left(-\csc\frac{\pi}{3}\cot\frac{\pi}{3}\right) + 80\left(\frac{\pi}{3}\right)^7\csc\frac{\pi}{3}\\ &= 10\left(\frac{\pi}{3}\right)^8\left(-\frac{2}{\sqrt{3}}\frac{1}{\sqrt{3}}\right) + 80\left(\frac{\pi}{3}\right)^7\frac{2}{\sqrt{3}}\\ &= -\frac{20\pi^8}{3^9} + \frac{160\pi^7}{3^7\sqrt{3}}.\end{aligned}$$

This is the slope of the line. And a point on the line is $\left(\frac{\pi}{3}, \frac{20\pi^8}{3^8\sqrt{3}}\right)$, which we get by plugging in $x = \frac{\pi}{3}$ into the original function. Now using point-slope form:

$$y = \underbrace{\left(-\frac{20\pi^8}{3^9} + \frac{160\pi^7}{3^7\sqrt{3}}\right)}_{m}\underbrace{\left(x - \frac{\pi}{3}\right)}_{x-x_1} + \underbrace{\frac{20\pi^8}{3^8\sqrt{3}}}_{y_1}.$$

□