Overview

In this lesson we finish our discussion on limits by learning how to find them by more analytical methods than just plugging in values or inspecting the graph.

Lesson

The idea of infinity can be quite illusive. Perhaps the most important fact at this level is that $\infty$ is not a number. As such we can not treat $\infty$ just like any real number that we’ve grown accustomed to. So when we write $\infty$ or $-\infty$, we should think of it as a limit of some function.

Indeterminate forms

As one might hope, $\infty + \infty = \infty$, $\infty \cdot \infty = \infty$ and $\infty \infty = \infty$. But what about $\infty - \infty$ or $0/0$ or $\infty/\infty$? Unfortunately, these forms have no universal answer, and we call them indeterminate forms. So when we get a limit of this form, the final answer could be $\infty$, 0, or something different altogether.

A recipe for finding limits

Let’s say we want to compute the limit $\lim_{x \to a} f(x)$. We proceed in the following fashion.

1. Try evaluating $f(a)$.
   
   (a) Did you get a (finite) number? Done. That’s the limit.
   
   (b) Did you get nonzero number? There are 3 possible outcomes: $+\infty$, $-\infty$, or the limit does not exist. Check the left- and right-hand limits to determine the final answer.
   
   (c) Did you get an indeterminate form? We must investigate further. Go to step 2.

2. Simplify algebraically as much as possible. Or multiply by a conjugate if appropriate.

3. Go back to step 1.

An interlude on conjugates

This doesn’t explicitly appear on homework 4, but it’s good to know anyway. Recall that the conjugate of $a + b$ is $a - b$. It has the convenient property that when we multiply a number by its conjugate we get a difference of squares:

\[ a^2 - b^2 = (a + b)(a - b). \]

In the context of limits, this will be useful when square roots appear. Say $a = \sqrt{x}$ and $b = \sqrt{y}$. Then the conjugate of $\sqrt{x} + \sqrt{y}$ is $\sqrt{x} - \sqrt{y}$. Moreover,

\[ (\sqrt{x} + \sqrt{y})(\sqrt{x} - \sqrt{y}) = \sqrt{x}\sqrt{x} + \sqrt{x}\sqrt{y} - \sqrt{x}\sqrt{y} - \sqrt{y}\sqrt{y} = x - y. \]

Example 1. Find the following limit if it exists.

\[ \lim_{x \to 0} \frac{\sqrt{x} + 2 - \sqrt{2}}{x} \]
Solution. Note that if we try to plug in \( x = 0 \), we get \( \frac{0}{0} \), which is no good. So here we use the method of multiplying the top and bottom by the conjugate of the numerator.

\[
\frac{\sqrt{x + 2} - \sqrt{2}}{x} = \frac{\sqrt{x + 2} - \sqrt{2}}{x} \cdot \frac{\sqrt{x + 2} + \sqrt{2}}{\sqrt{x + 2} + \sqrt{2}}
\]

\[
= \frac{(x + 2) - 2}{x(\sqrt{x + 2} + \sqrt{2})}
\]

\[
= \frac{x}{x(\sqrt{x + 2} + \sqrt{2})}
\]

\[
= \frac{1}{\sqrt{x + 2} + \sqrt{2}}
\]

Now we are free to take the limit:

\[
\lim_{x \to 0} \frac{1}{\sqrt{x + 2} + \sqrt{2}} = \frac{1}{\sqrt{0 + 2} + \sqrt{2}}
\]

\[
= \frac{1}{2\sqrt{2}}.
\]

Properties of limits

Suppose \( f \) and \( g \) are two functions with

\[
\lim_{x \to a} f(x) = L \quad \text{and} \quad \lim_{x \to a} g(x) = K.
\]

Then the following properties hold.

1. \( \lim_{x \to a} cf(x) = cL \), where \( c \) is a real number.

2. \( \lim_{x \to a} [f(x) \pm g(x)] = L \pm K \)

3. \( \lim_{x \to a} [f(x)g(x)] = LK \)

4. \( \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{K} \), as long as \( K \neq 0 \).

5. \( \lim_{x \to a} [f(x)]^n = L^n \), where \( n \) is a positive integer.

This amounts to saying that we can do pretty much all of the things that we would want to do with limits. If we have a sum/difference or a product/quotient that we want to take the limit of, we can take the limit of each part separately and then perform the indicated operation.

Examples

Let’s get some more practice computing limits using the tools above.

Example 2. Given the function

\[
f(x) = \frac{x^2 - 6x}{x^2 + 9x},
\]

find the following limits if they exist.
(a) \( \lim_{x \to 0} f(x) \)

(b) \( \lim_{x \to -9} f(x) \)

(c) \( \lim_{x \to 6} f(x) \)

**Solution.** (a) Using our recipe, the first thing we try is plugging in 0. But then we get \(\frac{0}{0}\), no good. Step 2 is to simplify algebraically:

\[
 f(x) = \frac{x^2 - 6x}{x^2 + 9x} = \frac{x(x - 6)}{x(x + 9)} = \frac{x - 6}{x + 9}. \tag{*}
\]

Now when we plug in 0, we get \(-\frac{6}{9} = -\frac{2}{3}\).

(b) If we plug in \(-9\) into our simplified version of \(f(x)\), we get \(-\frac{15}{0}\). According to 2(b) in our recipe we either have some type of \(\infty\) or DNE as our answer. To determine what type it is, we pick a number close to \(-9\) on either side and observe what happens.

<table>
<thead>
<tr>
<th>(x)</th>
<th>(-9.0001)</th>
<th>(-8.9999)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(f(x))</td>
<td>150,001</td>
<td>(-149,999)</td>
</tr>
</tbody>
</table>

This suggests that

\[
 \lim_{x \to -9^-} f(x) = +\infty \quad \text{but} \quad \lim_{x \to -9^+} f(x) = -\infty.
\]

Thus the limit does not exist.

(c) When we plug in 6 for \(x\) in our simplified version of \(f(x)\) we get \(\frac{0}{15} = 0\). \(\square\)

**Careful.** It’s tempting to get into the habit of identifying a fraction with 0 and another number with \(\infty\) or DNE, but remember that

\[
 \frac{0}{\text{any nonzero number}} = 0.
\]

**Remark.** Once you simplify a function algebraically, you should use the simplified version to compute all limits and to help you graph the function if needed.

**Example 3.** Given the function

\[
 f(x) = \begin{cases} 
 4x^2 + 8 & x \leq 0 \\
 19x + 8 & 0 < x < 1 \\
 -19x + 8 & x \geq 1 
\end{cases}
\]

find the following limits if they exist.

(a) \( \lim_{x \to 0} f(x) \)

(b) \( \lim_{x \to 1} f(x) \)
Remark. The function in this example is a piecewise function. Notice that each of the pieces are nice and we can evaluate them at any point we want. The potential problem is that we could get jumps when we piece these functions together.

In case you are unfamiliar with piecewise functions, all it means is the function is equal to the piece on the described interval. For example if we want to find \( f(-4) \), we would look at the first piece because \(-4 \leq 0\). If we wanted to find \( f(1/2) \), we would look at the second piece because \( 0 < 1/2 < 1 \). And if we wanted to look at \( f(10) \), we would look at the third piece because \( 10 \geq 1 \).

**Solution for Example 3.** (a) We want to be sure that the left- and right-hand limits agree. Let’s start with the left-hand limit. Notice that for \( x \leq 0 \), \( f(x) = 4x^2 + 8 \). So we can simply plug in \( x = 0 \) to find

\[
\lim_{x \to 0^-} 4x^2 + 8 = 0 + 8 = 8.
\]

Similarly, for the right-hand limit, we look at \( 19x + 8 \) because \( f(x) = 19x + 8 \) for \( 0 < x < 1 \). Again, we can plug in \( 0 \) here to get

\[
\lim_{x \to 0^+} 19x + 8 = 0 + 8 = 8.
\]

So the left- and right-hand limits agree and we have

\[
\lim_{x \to 0} f(x) = 8.
\]

(b) The process for finding the limit as \( x \to 1 \) is exactly the same. For values to the left of 1, we’re looking at the piece \( 19x + 8 \). So

\[
\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 19x + 8 = 19 + 8 = 27.
\]

And for the right-hand limit, we’re looking at the third piece.

\[
\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} -19x + 8 = -19 + 8 = -11.
\]

Notice that the left- and right-hand limits do not agree. So \( \lim_{x \to 1} f(x) \) does not exist.

**Example 4.** Given the function

\[
f(x) = \begin{cases} 
  x - 20 & x < 25 \\
  \sqrt{x} & x \geq 25 
\end{cases}
\]

find the limit \( \lim_{x \to 25} f(x) \) if it exists.

**Solution.** This is just a simpler example of a piecewise function. We compute the left- and right-hand limits:

\[
\lim_{x \to 25^-} f(x) = \lim_{x \to 25^-} (x - 20) = 5
\]

\[
\lim_{x \to 25^+} f(x) = \lim_{x \to 25^+} \sqrt{x} = 5
\]

The two limits agree. So the two-sided limit exists and is equal to 5 as well.
Example 5. Given the function

\[ f(x) = \begin{cases} 
  x - 20 & x \neq 25 \\
  15 & x = 25 
\end{cases}, \]

find the limit \( \lim_{x \to 25} f(x) \) if it exists.

Solution. If we wanted to we could rewrite our function like this

\[ f(x) = \begin{cases} 
  x - 20 & x < 25 \\
  15 & x = 25 \\
  x - 20 & x > 25 
\end{cases}, \]

to mirror the other types of piecewise functions. From here it should be clear that

\[ \lim_{x \to 25} f(x) = \lim_{x \to 25} (x - 20) = 5. \]

Notice that \( f(25) = 15 \neq 5 \), but that doesn’t matter to us when we’re just concerned about the limit. \( \square \)