Lesson 35

Eigenvalues & Eigenvectors

Recall: A vector is an $n \times 1$ matrix, usually denoted by u, v, w. e.g. $v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$. If we're dealing with $n=2$, we usually write $v = \begin{bmatrix} x \\ y \end{bmatrix}$, and for $n=3$, $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

Definition If A is an $n \times n$ matrix, λ a real number, and $Av = \lambda v$, then λ is called an eigenvalue, and v is called an eigenvector.

Finding eigenvalues

Since λ is a real number, let's use t as our variable. We're given $Av = tv$, ie, $Av = tIv$, where I is the identity matrix $\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$. Then we have

$$0 = tIv - Av$$

$$= (tI - A)v$$

(we have to factor out v on the right since order of matrix mult. matters.)

So $v = 0$ is clearly a solution. If there is another solution for $v \neq 0$, that means $tI - A$ is not invertible. (Otherwise we could multiply both sides by $(tI-A)^{-1}$ to get $v=0$.)

Recall A matrix is invertible if and only if its determinant is nonzero.

This means we want to solve for t in $\det(tI-A)=0$.

Definition $\det(tI-A)$ is the characteristic polynomial for A.

Ex1 Find the eigenvalues for the matrix $A=\begin{bmatrix}-4 & -2\\ 10 & 8\end{bmatrix}$.

Step 1 Figure out $tI-A$:

$$t\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}-\begin{bmatrix}-4 & -2\\ 10 & 8\end{bmatrix}=\begin{bmatrix}t & 0\\ 0 & t\end{bmatrix}-\begin{bmatrix}-4 & -2\\ 10 & 8\end{bmatrix}$$

$$=\begin{bmatrix}t+4 & 2\\ -10 & t-8\end{bmatrix}$$

Notice we will always get t minus the diagonal entries of A, and the negative of everywhere else.

Step 2 $0=\det(tI-A)=\begin{vmatrix}t+4 & 2\\ -10 & t-8\end{vmatrix}=(t+4)(t-8)+20$

$$=t^2-4t-32+20$$

$$=t^2-4t-12$$

$$=(t+2)(t-6)$$

This gives $\lambda_1=-2$, $\lambda_2=6$.

Finding eigenvectors

Could be asked to test whether given vectors are eigenvectors using $Av=\lambda v$.

Ex2 Which of the following vectors are eigenvectors for $\begin{bmatrix} -7 & 6 \\ -4 & 4 \end{bmatrix}$?

$\begin{bmatrix} 3 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 6 \\ 5 \end{bmatrix}$, $\begin{bmatrix} -3 \\ -4 \end{bmatrix}$

$$\begin{bmatrix} -7 & 6 \\ -4 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \end{bmatrix} \quad \text{no}$$

$$\begin{bmatrix} -7 & 6 \\ -4 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -8 \\ -4 \end{bmatrix} = -4\begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{yes}$$

$$\begin{bmatrix} -7 & 6 \\ -4 & 4 \end{bmatrix}\begin{bmatrix} 6 \\ 5 \end{bmatrix} = \begin{bmatrix} -12 \\ -4 \end{bmatrix} \quad \text{no}$$

$$\begin{bmatrix} -7 & 6 \\ -4 & 4 \end{bmatrix}\begin{bmatrix} -3 \\ -4 \end{bmatrix} = \begin{bmatrix} -3 \\ -4 \end{bmatrix} = 1 \cdot \begin{bmatrix} -3 \\ -4 \end{bmatrix} \quad \text{yes}$$

Could also be asked to find them yourself

Ex3 Find the eigenvalues and corresponding eigenvectors of $A = \begin{bmatrix} 0 & -2 \\ 5 & -7 \end{bmatrix}$

First find eigenvalues

Step 1 $\quad tI - A = \begin{bmatrix} t & 2 \\ -5 & t+7 \end{bmatrix}$

Step 2 $\quad 0 = \begin{vmatrix} t & 2 \\ -5 & t+7 \end{vmatrix} = t(t+7) + 10$

$$= t^2 + 7t + 10$$

$$= (t+2)(t+5)$$

So $\lambda_1 = -2$, $\lambda_2 = -5$ are our eigenvalues.

Next, find eigenvectors. Start with λ_1..

Step 1 Plug in λ_1 for t in $tI-A$: $\begin{bmatrix} -2 & 2 \\ -5 & 5 \end{bmatrix}$.

Step 2 Solve $\underbrace{\begin{bmatrix} -2 & 2 \\ -5 & 5 \end{bmatrix}}_{(\lambda_1 I - A)} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{v} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{0}$

Option 1 Row reduce $\left[\begin{array}{cc|c} -2 & 2 & 0 \\ -5 & 5 & 0 \end{array}\right] \xrightarrow{-\frac{1}{2}R_1} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ -5 & 5 & 0 \end{array}\right] \xrightarrow{5R_1+R_2} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

Option 2 Solve: $-2x + 2y = 0$ (1)
$-5x + 5y = 0$ (2)

(1) tells us $\boxed{x=y}$. Plug in x for y in (2) gives $0=0$.

Either way, tells us y is a "free variable." We can pick y and find an x using $x=y$. So pick $y=1$ (for simplicity).
Then $x=1$, and $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Now for λ_2

Step 1 $\begin{bmatrix} -5 & 2 \\ -5 & 2 \end{bmatrix}$

Step 2: $\left[\begin{array}{cc|c} -5 & 2 & 0 \\ -5 & 2 & 0 \end{array}\right] \xrightarrow{-R_1+R_2} \left[\begin{array}{cc|c} -5 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{-\frac{1}{5}R_1} \left[\begin{array}{cc|c} 1 & -2/5 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\Rightarrow$ y is free, $x = +\frac{2}{5}y$.

Pick $y=1$, then $x=\frac{2}{5}$.

So $v_2 = \begin{bmatrix} 2/5 \\ 1 \end{bmatrix}$.