Overview

By now we should be comfortable with all of the basic integrals mentioned in the previous lesson. But what about an integral like

$$\int e^{\tan x}\sec^2 x\,dx?$$

To be able to answer a question like this one, we turn to u-substitution. This is how we account for the chain rule when performing integration.

In the problem above, if we let $\boxed{u = \tan x,}$ then $\frac{du}{dx} = \sec^2 x$. After multiplying both sides by dx, we obtain $\boxed{du = \sec^2 x\,dx.}$ Now

$$\int \underbrace{e^{\tan x}}_{e^u}\underbrace{\sec^2 x\,dx}_{du} = \int e^u\,du.$$

Now this is a basic integral we know how to solve. Remembering that the antiderivative of e^u is e^u, we can say that

$$\int e^u\,du = e^u + C,$$

where C is an arbitrary constant. Finally, reusing the fact that we made the substitution $u = \tan x$, our final answer is

$$e^{\tan x} + C.$$

These problems can easily be made very complicated, but in this lesson, we will focus on somewhat simpler examples.

Examples

Example 1. Compute

$$\int 4x^6 e^{x^7}\,dx.$$

Solution. If we pick $u = x^7$, then $du = 7x^6\,dx$. This one doesn't line up quite as nicely as the previous problem, but that's okay. We start by factoring out the constants, giving

$$4\int x^6 e^{x^7}\,dx = e^{x^7}x^6\,dx.$$

Now there's no factor of 7 in the problem at hand, but note that $du = 7x^6\,dx$ implies that $\frac{1}{7}\,du = x^6\,dx$. Thus the integral becomes

$$\begin{aligned}\frac{4}{7}\int e^u\,du &= \frac{4}{7}e^u + C\\ &= \frac{4}{7}e^{x^7} + C.\end{aligned}$$

□

Example 2. Compute
$$\int x^2(5x^3+6)^{10}\,dx.$$

Solution. Let $u = 5x^3+6$. Then $du = 15x^2\,dx$. So $x^2\,dx = \frac{1}{15}\,du$. Now the integral becomes

$$\begin{aligned}\frac{1}{15}\int u^{10}\,du &= \frac{1}{11}\cdot\frac{1}{15}u^{11}+C\\ &= \frac{1}{165}(5x^3+6)^{11}+C\end{aligned}$$

□

Example 3. Compute
$$\int \frac{1}{\sqrt[5]{1-3t}}\,dt.$$

Solution. Let $u = 1-3t$. Then $du = -3\,dt$, so then $dt = -\frac{1}{3}\,du$. And our integral becomes

$$\begin{aligned}\int \frac{1}{\sqrt[5]{u}}\left(\frac{1}{3}\right)\,du &= -\frac{1}{3}\int u^{-1/5}\,du\\ &= -\frac{1}{3}\left(\frac{5}{4}u^{4/5}\right)+C\\ &= -\frac{5}{12}u^{4/5}+C.\end{aligned}$$

□

Example 4. Find a function f whose tangent line has the slope

$$\frac{2+\sqrt[3]{x}}{3\sqrt[3]{x^2}}$$

for all nonzero x and whose graph passes through the point $(1, \frac{5}{2})$.

Solution. First we need to find the functions whose tangent line has the given slope. This just means we need to compute the antiderivative. This also requires a u-substitution, so let $u = 2+\sqrt[3]{x}$. Then $du = \frac{1}{3}x^{-2/3}$. This may not seem helpful at first look, but if we rewrite the integral, it should become more apparent.

$$\begin{aligned}\int \frac{2+\sqrt[3]{x}}{3\sqrt[3]{x^2}}\,dx &= \int \left(2+\sqrt[3]{x}\right)\frac{1}{3\sqrt[3]{x^2}}\,dx \\ &= \int \left(2+\sqrt[3]{x}\right)\frac{1}{3}x^{-2/3}\,dx \\ &= \int u\,du \\ &= \frac{1}{2}u^2 + C \\ &= \frac{1}{2}\left(2+\sqrt[3]{x}\right)^2 - 2.\square\end{aligned}$$

Example 5. You arrive at 6 am to a crime scene and discover a body. Analysts have determined that in the present conditions the temperature of the body would decrease at a rate of

$$T'(t) = -11.82e^{-0.788t}$$

degrees Celsius per hour. The core temperature of the body at 6 am read to be 27°C. Assuming that the average person has a core temperature of 37°C, determine the time of death to the nearest minute.

Solution. Let's make the time of death our initial time, and we'll figure out how much time has passed since then. So then $T(0) = 37$, and $T'(t) = -11.82e^{-0.788t}$. Finding a general solution,

$$\begin{aligned}\int -11.82e^{-0.788t}\,dt &= \frac{-11.82}{-0.788}e^{-0.788t} \\ &= 15e^{-0.788t} + C.\end{aligned}$$

Using that $T(0) = 37$,

$$\begin{aligned}37 = T(0) &= 15e^{-0.788\cdot 0} + C \\ &= 15 + C,\end{aligned}$$

so $C = 22$.

Thus the temperature at any given time t hours after the initial time is given by

$$T(t) = 15e^{-0.788t} + 22.$$

Now we want to use the fact that presently the temperature is 27°C.

$$\begin{aligned} 27 &= 15e^{-0.788t} + 22 \\ 5 &= 15e^{-0.788t} \\ \frac{1}{3} &= e^{-0.788t} \\ \ln\left(\frac{1}{3}\right) &= e^{-0.788t} \\ \frac{\ln\left(\frac{1}{3}\right)}{-0.788} &= t \\ \boxed{1.39 \approx t.} \end{aligned}$$

Now recall that this means that we have discovered the body approximately 1.39 hours after the time of death. since 0.39 hours is approximately 23 minutes, the time of death is 1 hour and 23 minutes ago. That is, $\boxed{4:37\,\text{am.}}$ □

Example 6. Instead of living in a dorm, you decide to buy a house while at Purdue. You find one appraised at $60,000, and according to latest market trends, the value of the house will increase at a rate of

$$H'(t) = \frac{2.5t^4}{\sqrt{0.5t^5 + 6400}}$$

dollars per year. What will the house be worth if you sell it after graduating? (You may assume it only takes 4 years to graduate.)

Solution. First we must determine what the value of the house is at any given time t years from now. To this, we integrate,

$$\begin{aligned} \int H'(t)\,dt &= \int \frac{2.5t^4}{\sqrt{0.5t^5 + 6400}}\,dt \\ &= \int \frac{du}{\sqrt{u}} && \boxed{u = 0.5t^5 + 6400} \\ &= \int u^{-1/2}\,du && \boxed{du = 2.5t^4\,dx} \\ &= 2u^{1/2} + C \\ &= 2\sqrt{0.5t^5 + 6400} + C \end{aligned}$$

Now we use the fact that $H(0) = 60,000$.

$$\begin{aligned} H(0) = 60000 &= 2\sqrt{0.5(0)^5 + 6400} + C \\ &= 2\sqrt{6400} + C = 2 \cdot 80 + C \\ &= 160 + C. \end{aligned}$$

This tells us that $C = 60000 - 160 = 59840$. So our equation for H is

$$H(t) = 2\sqrt{0.5t^5 + 6400} + 59840.$$

Then four years from now, the house will be worth

$$\begin{aligned} H(4) &= 2\sqrt{0.5(4)^5 + 6400} + 59840 \\ &= 60006.28. \end{aligned}$$

So the house would be worth \$60,006.28. So it's not a particularly lucrative investment, but at least you would come out ahead if you could sell it at market value. □