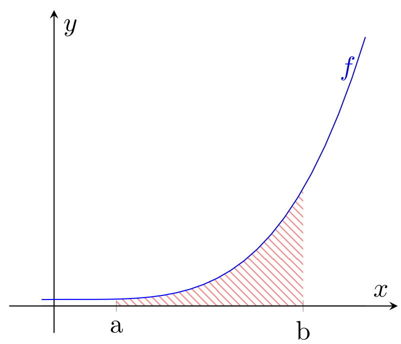
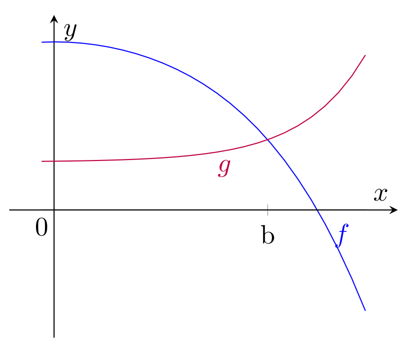
Overview

In this lesson we return to standard Calculus II material with areas between curves. Recall from first semester calculus that the definite integral had a geometric meaning, namely the area under a curve. The red region in the graph below is given by $\int_a^b f(x)\,dx$.

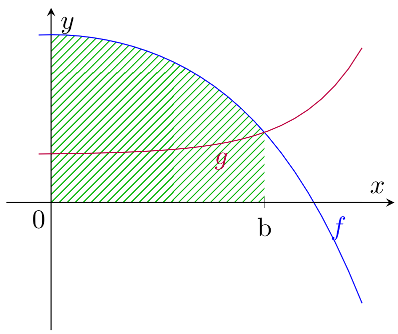


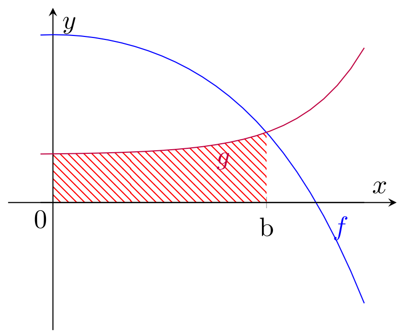
Lesson

In this lesson we're concerned with a slightly different problem, namely the area between two curves. For example, consider the graphs of f and g below, and let's say we want to calculate the area bounded by the two curves between $x = 0$ and $x = b$.

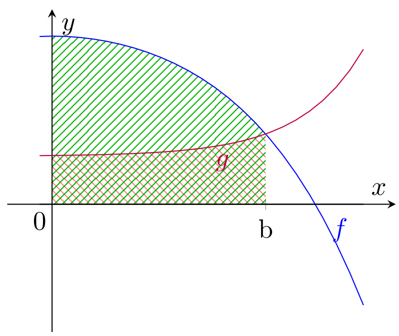


If we calculate the area under each curve separately, we find the green and red areas in the two graphs below.

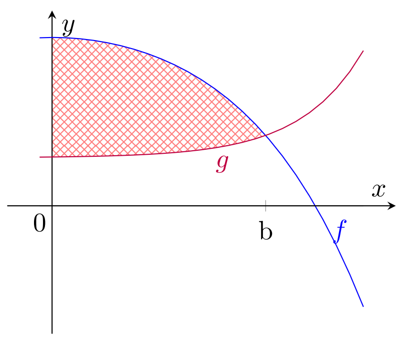




Laying one on top of the other it's pretty clear (if it wasn't already) that the area between the curve g and the x-axis is counted twice.



So if we take the green area and subtract the red area we precisely get the area between the two curves.



Using integrals to calculate, we have

$$\begin{aligned}\text{Area} &= \int_0^b f(x)\,dx - \int_0^b g(x)\,dx \\ &= \int_0^b \left(f(x) - g(x)\right)\,dx.\end{aligned}$$

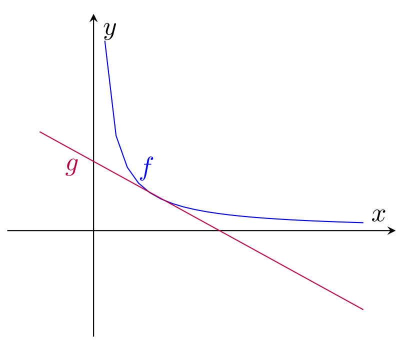
More generally, if we want to find the area between two curves on an interval $[a, b]$, we do this by computing

$$\begin{aligned}\text{Area} &= \int_a^b \text{Top} - \int_a^b \text{Bottom} \\ &= \int_a^b (f(x) - g(x))\, dx\end{aligned}$$

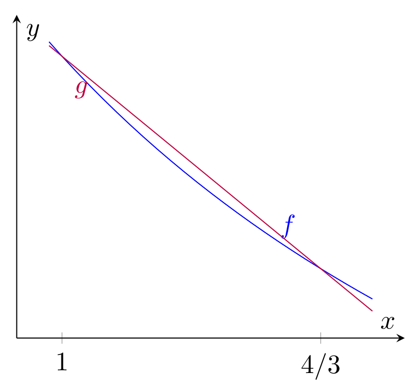
Example 1. Find the area of the region bounded by the curves

$$f(x) = \frac{12}{x} \quad \text{and} \quad g(x) = -9x + 21.$$

Solution. We want to integrate the "top" minus the "bottom." So it would be constructive (as with many integral problems) to graph f and g. The big picture graph looks like this.



But that's hard to see, so let's zoom in.



Now we can see that f starts out above g, then dips below before coming back up again. So for the area between the two, g is the top function. So we want

$$\int_a^b \left(-9x + 21 - \frac{12}{x}\right) dx.$$

But what are a and b? Points of intersection occur when the two functions are equal. So setting $f(x) = g(x)$,

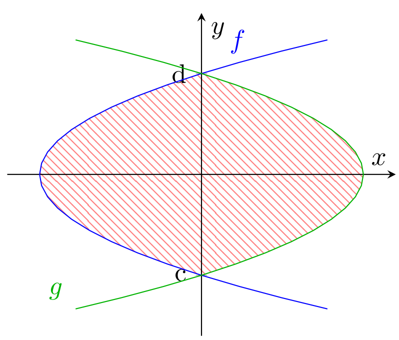
$$\begin{aligned} \frac{12}{x} &= -9x + 21 \\ 12 &= -9x^2 + 21x \\ 0 &= 9x^2 - 21x + 12 \\ 0 &= 3(3x^2 - 7x + 4) \\ 0 &= 3(3x-4)(x-1), \end{aligned}$$

which has solutions of $x = 1$ and $x = \frac{4}{3}$. So there's our a and b. Now only to calculate

$$\begin{aligned} \int_1^{4/3} \left(-9x + 21 - \frac{12}{x}\right) dx &= \frac{-9x^2}{2} + 21x - 12\ln x \bigg|_1^{4/3} \\ &= \frac{7}{2} - 12\ln\left(\frac{4}{3}\right). \end{aligned}$$ □

Integrating with respect to x is nothing special, it's just what we usually do. Some problems will lend themselves well to integrating with respect to y instead, and for others the only way to solve the problem will be integrating with respect to y.

How does this change the problem at hand? The roles of x and y switch. Given two curves f and g which are functions of y, we want to integrate the one with larger x values minus the one with the smaller x values.



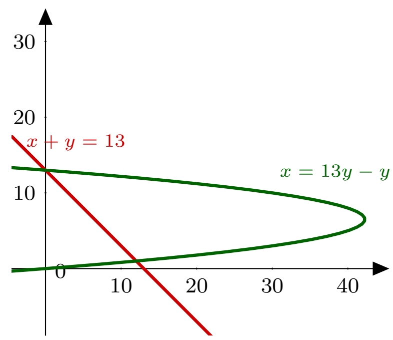
So we could amend our previous "Top minus Bottom" to "Bigger minus Smaller." Specifically for functions of y, we have

$$\begin{aligned} \text{Area} &= \int_c^d \text{Right} - \int_c^d \text{Left} \\ &= \int_c^d (F(y) - G(y))\, dy \end{aligned}$$

Example 2. Find the area bounded by the curves

$$x = 13y - y^2 \quad \text{and} \quad x + y = 13.$$

Solution. The graphs in this example are a bit easier to draw. If drawing the parabola gives you trouble, try graphing it as if it were $y = 13x - x^2$ then flipping your paper along the $y = x$ line.



To figure out where the two functions intersect, we set them equal to each other. We were given the line as the equation $x + y = 13$, so first we want to solve this for x: $x = 13 - y$. Now

$$\begin{aligned} 13 - y &= 13y - y^2 \\ 0 &= y^2 - 14y + 13 \\ 0 &= (y - 13)(y - 1). \end{aligned}$$

So the y-intercepts are $y = 1, 13$. These are the bounds for our integral, and now we use "Right minus left."

$$\begin{aligned} &\int_1^{13} \left[(13y - y^2) - (13 - y)\right] \, dy \\ &= \int_1^{13} \left[-y^2 + 14y - 13\right] \, dy \\ &= -\frac{1}{3}y^3 + 7y^2 - 13y\Big|_1^{13} \\ &= \left[-\frac{1}{3}(13)^3 + y(13)^2 - 13(13)\right] - \left[-\frac{1}{3} + 7 - 13\right] \\ &= 288 \end{aligned}$$

□

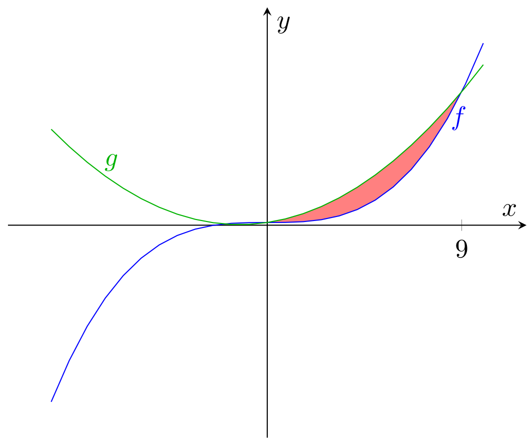
Example 3. Find the area bounded by the curves

$$y = x^3 + 15 \quad \text{and} \quad y = 7x^2 + 18x + 15 \quad \text{where } x \geq 0.$$

Solution. Let's call the cubic one f and the quadratic g. To find where f and g intersect, we set them equal to each other.

$$\begin{aligned} x^3 + 15 &= 7x^2 + 18x + 15 \\ 0 &= x^3 - 7x^2 - 18x \\ x(x+2)(x-9). \end{aligned}$$

So since we are sticking with $x \geq 0$, the points of intersection are $x = 0, 9$. Now graphing the two polynomials should be a breeze.



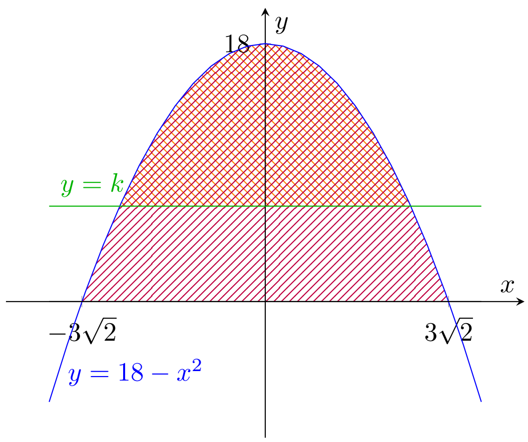
Now to calculate the area between g and f,

$$\begin{aligned} \int_0^9 (g - f)\,dx &= \int_0^9 \left[(7x^2 + 18x + 15) - (x^3 - 15)\right]\,dx \\ &= \int_0^9 \left[-x^3 + 7x^2 + 18x\right]\,dx \\ &= -\frac{x^4}{4} + \frac{7}{3}x^3 + 9x^2\Big|_0^9 \\ &= -\frac{1}{4}(9)^4 + \frac{7}{3}(9)^3 + 9(9)^2 \\ &= 789.75 \end{aligned}$$

□

Example 4. Find the equation of the horizontal line that divides the area of the region bounded by $y = 18 - x^2$ and $y = 0$ in half.

Solution. We definitely want to graph this one. We have a parabola with vertex at $(0, 18)$ and the zeros are at $x = \pm\sqrt{18} = \pm 3\sqrt{2}$. Let's call the horizontal line we're after $y = k$.



So we want the area in the each of those two areas to be equal to half the area between $y = 18 - x^2$ and $y = 0$. That means we should first figure out what the total area:

$$\begin{aligned}\int_{-3\sqrt{2}}^{3\sqrt{2}} (18 - x^2)\,dx &= 2\int_0^{3\sqrt{2}} (18 - x^2)\,dx \\ &= 18x - \frac{1}{3}x^3\Big|_0^{3\sqrt{2}} \\ &= 18 \cdot 3\sqrt{2} - \frac{1}{3}(3\sqrt{2})^3 \\ &= 72\sqrt{2}.\end{aligned}$$

Note. In the first line we have used that $18 - x^2$ is symmetric about the y-axis. That means that whenever we plug in x or $-x$ we get the same value. You could have skipped that step and just evaluated the integral normally.

Thus we want the area between $y = 18 - x^2$ and $y = k$ to be equal to $\frac{72}{2}\sqrt{2} = 36\sqrt{2}$. One way we could do this is to make up an integral in terms of y. If $y = 18 - x^2$ then $x = \pm\sqrt{18 - y}$. So we could think of the parabola as two functions of y put together. So the area is given by "Right minus Left." The bounds are easily seen to be $y = k$ and $y = 18$. Now

$$\begin{aligned}
36\sqrt{2} &= \int_k^{18} \left[\sqrt{18-y} - (-\sqrt{18-y})\right]\, dy \\
&= 2\int_k^{18} \sqrt{18-y}\, dy \\
&= -2\int_{18-k}^{0} u^{1/2}\, du && \begin{aligned} u &= 18-y \\ du &= -dy \end{aligned} \\
&= 2\int_0^{18-k} u^{1/2}\, du \\
&= 2\cdot\frac{2}{3}u^{3/2}\Big|_0^{18-k} \\
&= \frac{4}{3}(18-k)^{3/2}
\end{aligned}$$

At this point we just need to do a little algebra to solve for k:

$$\begin{aligned}
36\sqrt{2} &= \frac{4}{3}(18-k)^{3/2} \\
\frac{3}{4}\cdot 36\sqrt{2} &= (18-k)^{3/2} \\
27\sqrt{2} &= (18-k)^{3/2} \\
\left(27\sqrt{2}\right)^{2/3} &= 18-k \\
k &= 18 - \left(27\sqrt{2}\right)^{2/3} \\
k &\approx 6.66.
\end{aligned}$$

□

Remark. We could have done this last part as an integral in terms of x. In that case we would have had

$$\int_a^b \left[(18-x^2) - k\right]\, dx,$$

where a and b are the x values where the parabola and line intersect. How do we find those points? Well in the worked solution we solved the parabolic equation for x. We know that the y value is equal to k where they intersect, so the x-values are

$$x = \pm\sqrt{18-k}.$$

Then the integral we would set up would be

$$\int_{-\sqrt{18-k}}^{\sqrt{18-k}} \left[18 - x^2 - k\right]\, dx.$$

If you compute this, you should still get $\frac{4}{3}(18-k)^{3/2}$, then the rest is the same as before.

Remark. Although we haven't covered any word problems, homework 11 has a few, but they are pretty straightforward applications. It's worth recalling a few definitions though. In the context of economics, *revenue* is the total money brought in, *cost* has the obvious definition, and *profit* is revenue minus cost.

We should further recall that any time we are given a rate of *something* per *something else*, computing the definite integral gives us an answer in *something*s.