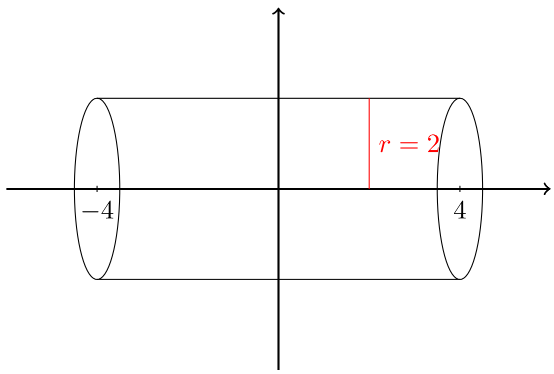
Overview

Now that we're experts in finding areas between curves, the next thing we want to tackle is volumes. Specifically, volumes of revolution. This is where we take some region in the plane and revolve it about an axis, and the question is what is the volume of the resulting solid. In this lesson we will only concern ourselves with regions which are revolved about the x- or y-axes.

Lesson

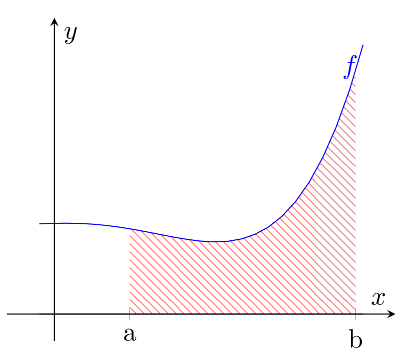
Say we want to calculate the volume of the cylinder below.

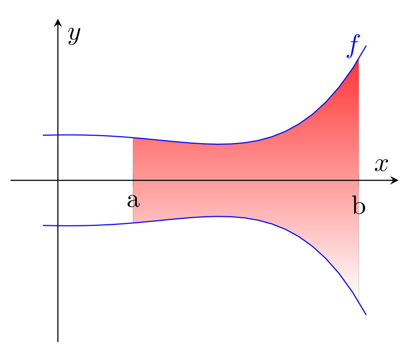


We know how to do this without calculus. (Or we can look it up quickly.) The volume of a cylinder of height 8 and radius 4 is $\pi r^2 h = 32\pi$. One way we could think about this is to take $\pi r^2 = 4\pi$ and integrate.

$$\int_{-4}^{4} 4\pi \, dx = 32\pi.$$

This may seem a bit contrived, but this is precisely how we want to deal with solids of revolution in general. Given any curve that's a function of x, say $f(x)$ and an interval $[a, b]$, we wish to find the volume that results by rotating that region about the x-axis.





So to find the volume of the solid obtained by rotating $f(x)$ about the x-axis on the interval $a \leq x \leq b$ is given by

$$\text{Volume} = \pi \int_a^b [f(x)]^2 \, dx$$

Given a function $g(y)$ we can compute the volume obtained by rotating g about the y-axis on the interval $c \leq y \leq d$ in a similar way:

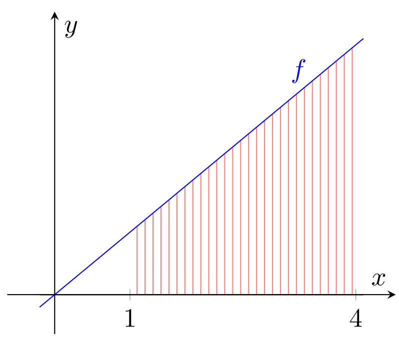
$$\text{Volume} = \int_c^d [g(y)]^2 \, dy$$

Example 1. Find the volume of the solid that results by revolving the region enclosed by the curves

$$y = 5x, \qquad x = 1, \qquad x = 4, \qquad y = 0$$

about the x-axis.

Solution. We want to rotate the following region about the x-axis.



Since our radii in the picture extend from the x axis to the y-value of the function, we should be thinking "dx."

$$\begin{aligned} \pi \int_1^4 (5x)^2 \, dx &= 25\pi \int_1^4 x^2 \, dx \\ &= 25\pi \left. \frac{1}{3} x^3 \right|_1^4 \\ &= \frac{25\pi}{3}(4^3 - 1) \\ &= 525\pi. \end{aligned}$$

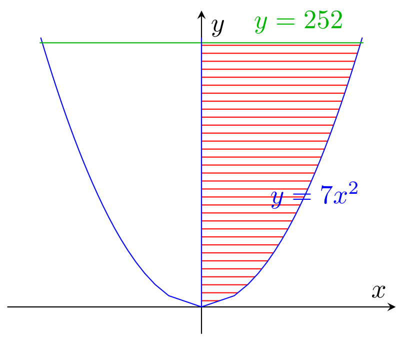
□

Example 2. Find the volume of the solid that results by revolving the region in the first quadrant enclosed by the curves

$$y = 7x^2, \qquad x = 0, \qquad y = 252$$

about the y-axis.

Solution. We want to rotate the following region about the y-axis.



Here, our radii are going from the y-axis to the x-value of the function. So in this case we should be thinking "dy."

$$\begin{aligned} \pi \int_0^{252} \left(\sqrt{\frac{y}{7}}\right)^2 dy &= \pi \int_0^{252} \frac{y}{7}\, dy \\ &= \pi \cdot \frac{1}{14} y^2 \Big|_0^{252} \\ &= 4536\pi. \end{aligned}$$

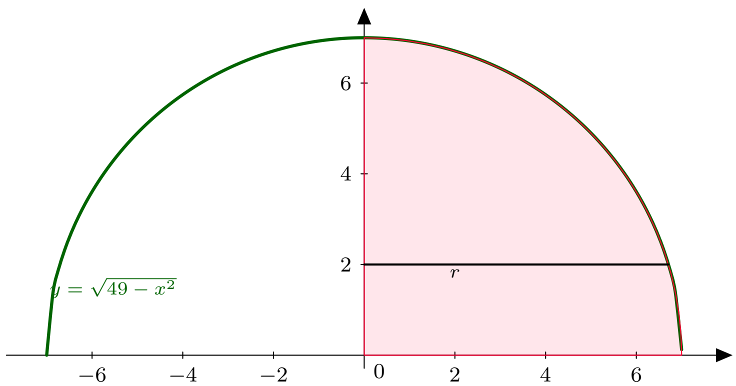
□

Example 3. Find the volume of the solid that results by revolving the region enclosed by the curves

$$y = \sqrt{49 - x^2}, \qquad y = 0, \qquad x = 0$$

about the y-axis.

Solution. If we square both sides of $y = \sqrt{49 - x^2}$, we should see that the function we are given is the upper half-circle of radius 7.



Here our radii are coming out of the y-axis, so we should be thinking "dy." So we just solve $y = \sqrt{49 - x^2}$ for x. We only need to take the positive one since we're working with the first quadrant.

$$\begin{aligned}
\pi \int_0^7 \left(\sqrt{49 - y^2}\right)^2 \, dy &= \pi \int_0^7 \left(49 - y^2\right) \, dy \\
&= \pi \left(49y - \frac{1}{3}y^3\right)\bigg|_0^7 \\
&= \pi(7^3 - \frac{1}{3}7^3) \\
&= \frac{2}{3} \cdot 7^3 \pi.
\end{aligned}$$

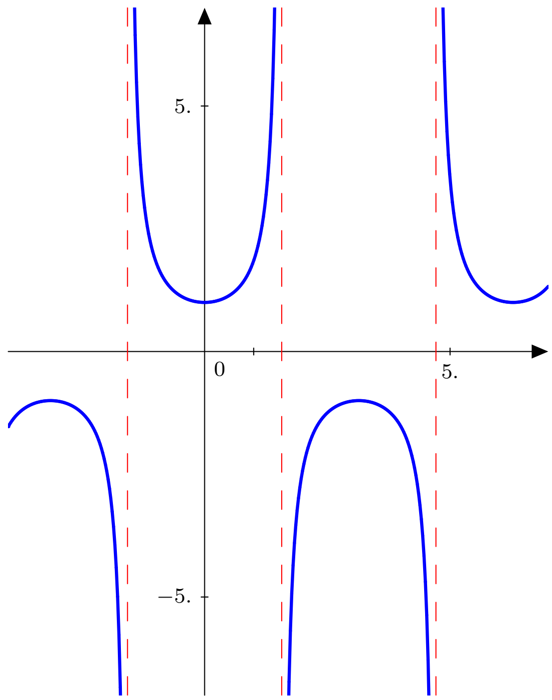
Notice that this is $\frac{1}{2}(\frac{4}{3}\pi r^3)$, half the volume of a sphere of radius 7. □

Example 4. Find the volume of the solid that results by revolving the region enclosed by the curves

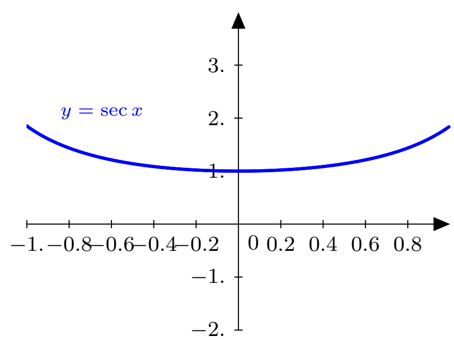
$$y = \sec x, \qquad y = 0, \qquad x = 0, \qquad x = \frac{\pi}{11}$$

about the x-axis.

Solution. Figuring out how to draw the graph of $y = \sec x$ may be a little tricky at first. Note that $\sec x = 1/\cos x$. So we are going to have vertical asymptotes at $x = \frac{\pi}{2} + n\pi$, where n is an integer. In the graph below, we have depicted the vertical asymptotes at $x = -\pi/2, \pi/2$, and $3\pi/2$ respectively. Then it would be good to plot a few extra points. When $x = 0 + 2n\pi$ (n an integer), $\sec x = 1$, and when $x = \pi + 2n\pi$, $\sec x = -1$. From there, using our knowledge of vertical asymptotes should get us a good sketch.



We are interested in rotating the region in the first quadrant bounded by $y = \sec x$ and $x = \frac{\pi}{11}$. Since $\frac{\pi}{11} < \frac{\pi}{2}$, we don't have to worry about any vertical asymptotes. If we zoom in to that picture, it looks something like this.



Now we proceed in the usual way.

$$\begin{aligned}\pi \int_0^{\pi/11} \sec^2 x \, dx &= \pi \tan x \Big|_0^{\pi/11} \\ &= \pi \left(\tan\left(\frac{\pi}{11}\right) - \tan 0 \right) \\ &= \pi \tan\left(\frac{\pi}{11}\right). \end{aligned}$$ □