Overview

In this section we discuss series; in particular, geometric series are of interest as they are far-reaching in application and frankly easy to calculate.

Lesson

First there is a bit of terminology to get out of the way. One word that will get thrown around a lot is *sequence*, which is precisely what you think it is: a list of numbers in a particular order. For example, $0, 2, 4, 6, \ldots$ is a sequence of nonnegative even integers. Of course, a sequence could be infinite or finite; the more interesting case is if we have an infinite sequence.

Definition 1. Given an infinite sequence, $a_0, a_1, a_2, \ldots$, of numbers, if we add all of the numbers in the sequence together, this is called a *series* (sometimes *infinite series*), and we write

$$\boxed{a_0 + a_1 + a_2 + \cdots = \sum_{n=0}^{\infty} a_n}$$

When we express a series as in the right hand side of the equation, we say that we are expressing the series in *sigma* or *summation notation*. (Σ is the Greek letter capital sigma.)

Definition 2. If we look at just the first n terms in the series, this is called the *nth partial sum*, and we write

$$s_n = \sum_{i=0}^{n-1} a_i.$$

The series is said to be *convergent* if $\lim_{n\to\infty} s_n$ exists and is equal to a finite real number, say s. If $\lim_{n\to\infty} s_n$ is infinite or does not exist, then the series is said to be *divergent*.

That's a lot of definitions at once, so let's look at some examples to put the terminology into practice.

Example 1. Find the 4th partial sum in the series

$$\sum_{n=1}^{\infty} \frac{3}{2} n^2$$

Solution. For $n = 1, 2, 3, 4$, we find

n	a_n
1	$3/2$
2	$(3/2) \cdot 2^2$
3	$(3/2) \cdot 3^2$
4	$(3/2) \cdot 4^2$

So $s_4 = \frac{3}{2}(1 + 2^2 + 3^2 + 4^2) = 45$. $\square$

Example 2. Write the series in sigma notation.

$$\frac{2}{3} + \frac{4}{6} + \frac{8}{18} + \frac{16}{72} + \frac{32}{360}$$

Solution. First, looking at the numerators, we see $2, 4, 8, 16, 32, \ldots$. If we start our series at $n = 1$, it is easy to see that the general a_n will have 2^n in the numerator. The denominator is a bit more tricky. But you may notice that all of the denominators are divisible by 3. So if we factor our a 3 from each of the denominators, we produce the sequence of denominators

$$3, 3 \cdot 2, 3 \cdot 6, 3 \cdot 24, 3 \cdot 120, \ldots$$

Looking at the leftovers, we see $1, 2, 6, 24, 120, \ldots$ This may seem a bit unfamiliar, but we could write this as

$$1, 1 \cdot 2, 1 \cdot 2 \cdot 3, 1 \cdot 2 \cdot 3 \cdot 4, 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5, \ldots$$

And this is precisely the factorial ($n!$). So the denominator of the general a_n is $3(n!)$. Finally, putting this together in sigma notation,

$$\sum_{n=1}^{\infty} \frac{2^n}{3(n!)}. \qquad \square$$

Example 3. Write the series using summation notation.

$$9 - \frac{18}{8} + \frac{27}{27} - \frac{36}{64} + \frac{45}{125}$$

Solution. Again we'll start our series from $n = 1$. Starting with the numerators we have $9, -18, 27, -36, 45, \ldots$ Let's start with the alternating sign: $+, -, +, \ldots$ How can we represent this for the general a_n? If $a_1 = 9, a_2 = -\frac{18}{8}$, etc., then for n odd, a_n is positive, and for n even, a_n is negative. The easiest way to represent this is $(-1)^{n+1}$. Next, we see that the numerators are multiples of 9. Thus so far we know

$$a_n = \frac{(-1)^{n+1} \cdot 9n}{\text{something}}.$$

Now how about the denominators? $1, 8, 27, 64, 125, \ldots$ If we think for a minute we'll see that each one of these is just n^3. So putting this all together in summation notation, we have

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cdot 9n}{n^3}. \qquad \square$$

Remark. It can be quite a laborious task to try to come up with the general a_n term in the series. Fortunately, in the Lon-Capa homework, these types of questions are multiple choice. So you can write out the first few terms in each of the answer choices and determine which choice matches the given problem.

Note. You may have noticed by this point we have looked at series with two different starting points, namely $n = 0$ and $n = 1$. In our definition we used $n = 0$, but really a series can have any starting value. This is something to look out for and can lead to representing the same series in two different ways.

Question. Which is greater: $0.\bar{9}$ or 1?

At first glance, this may seem like a ridiculous question, and the fact that it's even a question may lead you to believe that the answer is counterintuitive. To answer it we'll look into *geometric series*. A geometric series is a series where all the terms have a common ratio, r, and is of the form

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \cdots$$

As we mentioned before, one thing that makes this series special is we know how to calculate it. To do this let's look at the nth partial sum, s_n, and do some algebra trickery:

$$\begin{array}{rl} s_n = & a + ar + ar^2 + \cdots + ar^n \\ -rs_n = & \quad - ar - ar^2 - \cdots + ar^n - ar^{n+1} \\ \hline s_n - rs_n = & a \qquad\qquad\qquad\qquad\qquad - ar^{n+1} \end{array}$$

Now solving for s_n,

$$\begin{aligned} s_n - rs_n &= a - ar^{n+1} \\ s_n(1-r) &= a(1 - r^{n+1}) \\ s_n &= \frac{a(1-r^{n+1})}{1-r}. \end{aligned}$$

Finally, recall that the value of the sum is $\lim_{n\to\infty} s_n$, and if $-1 < r < 1$,

$$\begin{aligned} \lim_{n\to\infty} s_n &= \lim_{n\to\infty} \frac{a(1-r^{n+1})}{1-r} \\ &= \frac{a(1 - \lim_{n\to\infty} r^{n+1})}{1-r} \\ &= \frac{a}{1-r}, \end{aligned}$$

where in the last line we've used that if $-1 < r < 1$, then $r^{n+1} \to 0$ as $n \to \infty$. Thus the geometric series converges if $|r| < 1$ and diverges if $|r| > 1$, and this leads to the important formula

$$\boxed{\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, \quad |r| < 1} \tag{1}$$

Note. This formula will be (and has been) provided on each of the midterms, so you won't have to memorize the formula in and of itself. You will, however, have to know how to use it.

With these tools, we are now ready to answer the big question.

Example 4. Write $0.\bar{9}$ as a series in sigma notation and compute the series.

Solution. Recall that $0.\bar{9} = 0.999999\ldots$. Another way we could write this is

$$0.9 + 0.09 + 0.009 + 0.0009 + \cdots$$

And yet another way we could write this is

$$\frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots$$

Factoring out a $\frac{9}{10}$,

$$\frac{9}{10}\left(1 + \frac{1}{10} + \frac{1}{100} + \cdots\right)$$

Now the stuff in the parentheses mirrors our geometric series formula with $r = \frac{1}{10}$. So in sigma notation, we have

$$\sum_{n=0}^{\infty} \frac{9}{10}\left(\frac{1}{10}\right)^n.$$

To compute, we simply use (1):

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{9}{10}\left(\frac{1}{10}\right)^n &= \frac{9}{10} \cdot \frac{1}{1 - \frac{1}{10}} \\ &= \frac{9}{10} \cdot \frac{1}{\frac{9}{10}} \\ &= 1.\end{aligned}$$

This means that $0.\bar{9}$ and 1 are actually equal! $\square$

Example 5. Compute

$$\sum_{n=1}^{\infty} \left(\frac{6}{14}\right)^n$$

Solution. Here $r = \frac{6}{14}$, but notice that we are starting the series at $n = 1$. One way to do this is to shift the index from starting at $n = 1$ to starting at $n = 0$. To do this, notice that the first term in the series is $\frac{6}{14}$, so when we start from $n = 0$, the first term should still be $\frac{6}{14}$. Looking at $\sum_{n=0}^{\infty} \frac{6}{14}$, we see that the *second* term is $\frac{6}{14}$. So we can shift the index as follows:

$$\begin{aligned}
\sum_{n=1}^{\infty}\left(\frac{6}{14}\right)^{n} &= \sum_{n=0}^{\infty}\left(\frac{6}{14}\right)^{n+1} \\
&= \sum_{n=0}\frac{6}{14}\left(\frac{6}{14}\right)^{n} \\
&= \frac{\frac{6}{14}}{1-\frac{6}{14}} \qquad (2) \\
&= \frac{\frac{6}{14}}{\frac{14}{14}-\frac{6}{14}} \\
&= \frac{\frac{6}{\cancel{14}}}{\frac{8}{\cancel{14}}} \\
&= \frac{6}{8} \\
&= \frac{3}{4}.
\end{aligned}$$

□

The value of the method used in Example 5 is that it allows us to memorize only the one formula. The drawback is that it requires a little extra algebraic manipulation that some may be uncomfortable with. One way to deal with this is to come up with a formula for geometric series starting at $n = 1$. You'll notice that (2) is quite similar to (1), just with an extra $\frac{6}{14}$. So for a geometric series starting at $n = 1$,

$$\boxed{\sum_{n=1}^{\infty} ar^n = \frac{ar}{1-r}, \quad |r| < 1} \qquad (3)$$

Example 6. Compute

$$\sum_{n=1}^{\infty}\frac{3(-1)^n}{8^{2n}}$$

Solution. Here it might be a little less clear that this is a geometric series. But after some manipulating, it should be clear.

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{3(-1)^n}{8^{2n}} &= \sum_{n=1}^{\infty}\frac{3(-1)^n}{(8^2)^n} \\
&= \sum_{n=1}^{\infty} 3\left(\frac{-1}{8^2}\right)^n \\
&= \sum_{n=1}^{\infty} 3\left(\frac{-1}{64}\right)^n
\end{aligned}$$

Now it should be clear that this is a geometric series with $a = 3$ and $r = -\frac{1}{64}$. Note that this series also starts at $n = 1$. Using equation (3) for geometric series starting at $n = 1$, we easily compute

$$\begin{aligned}
\sum_{n=1}^{\infty} 3\left(\frac{-1}{64}\right)^n &= 3 \cdot \frac{-\frac{1}{64}}{1-(-\frac{1}{64})} \\
&= 3 \cdot \frac{-\frac{1}{64}}{\frac{64}{64}+\frac{1}{64}} \\
&= 3 \cdot \frac{-\frac{1}{\cancel{64}}}{\frac{65}{\cancel{64}}} \\
&= 3\left(-\frac{1}{65}\right) \\
&= -\frac{3}{65}.
\end{aligned}$$

$\square$

Although it didn't come up explicitly in these examples, we'll end with a useful fact.

Useful Fact. Given convergent series $\sum a_n$ and b_n and any number c,

1. $\sum(a_n + b_n) = \sum a_n + \sum b_n$
2. $\sum(a_n - b_n) = \sum a_n - \sum b_n$
3. $\sum ca_n = c\sum a_n$

In the useful fact, we didn't write the limits on the series, and this is often done when it is clear from context or it is immaterial. We do that here for the latter reason. However when writing series in sigma notation for homework, quizzes, etc., you should take care to write where the start from.