# Overview

In this lesson we move on from Calculus II material to multivariate calculus. Up to now, we've only considered functions that have one numerical input, and one numerical output. But many problems require more than one variable. And that's where we begin.

# Lesson

For functions of a single variable, we would write $y = f(x)$, $y$ is a function of $x$. For example $f(x) = \sqrt{1-x^2}$ is a one-variable function. But if both $x$ and $y$ are variables, then we can write a function $z = f(x, y)$, meaning that $z$ is a function of both $x$ and $y$. An example would be $f(x, y) = \sqrt{1 - x^2 - y^2}$. What does the graph of this function look like? It happens to be the upper half sphere of radius 1. Notice that the $f(x)$ we gave was the upper half circle of radius 1. Our knowledge of single-variable calculus will help us greatly to understand calculus of more than one variable. While there is no limit on the number of variables we can have in a function, we will restrict ourselves to functions of two variables.

It can be easy to get lost in the added complexity of multiple variables, but many things stay the same. If we want to compute the value of a two-variable function at a point, we will need an ordered pair $(x, y)$ instead of just an $x$-value.

**Example 1.** Find $f(1, -6)$, where

$$f(x,y) = \frac{-x+y}{-3x+10y}.$$

*Solution.* We simply plug in 1 for $x$ and $-6$ for $y$:

$$f(1,-6) = \frac{-1+(-6)}{-3(1)+10(-6)} = \frac{-7}{-63} = \frac{1}{9}$$ □

After getting the hang of what a function is in one variable, the next thing one usually does is study domain. Recall that the *domain* of a function is the allowable input. With two variables, we need to be a little more careful, but the idea is exactly the same. It may be helpful to review the domain of basic functions, like the square root, logarithm, rational functions, exponential functions, trig, etc.

In order to describe the domain of functions in two variables, we like to use set notation. A set is just a collection of some objects, and we denote sets with the curly braces, $\{\text{things in the set}\}$. If we want to specify conditions for being in the set, we use a colon or vertical bar to separate the type of things in the set and the conditions for being in the set. That is,

$$\{\text{things}\colon \text{some conditions}\},$$

which would be read as "the set of things such that those things satisfy some conditions." For example, we could write the first quadrant as a set as

$$\left\{(x, y) \in \mathbb{R}^2 \colon x \geq 0,\, y \geq 0\right\}.$$

The symbol $\in$ is read "in," and $\mathbb{R}^2$ denotes the $xy$-plane. So we would read this set as "the set of ordered pairs $(x, y)$ such that $x \geq 0$ and $y \geq 0$."

**Example 2.** Find the domain of

$$f(x, y) = \frac{-8e^{xy}}{17 - e^{xy}}.$$

*Solution.* The only domain issue for this one is if the denominator is equal to 0, so we can't have $17 - e^{xy} = 0$. In set notation, the domain is

$$\left\{(x, y) \colon e^{xy} \neq 17\right\}. \qquad \square$$

**Example 3.** Find the domain of

$$f(x, y) = \frac{6x}{\ln(10x + 10y)}.$$

*Solution.* Here there are two domain issues. We can't have the denominator be 0, and the argument of ln must be positive. So the domain is

$$\left\{(x, y) \colon 10x + 10y \neq 1,\, 10x + 10y > 0\right\}. \qquad \square$$

**Example 4.** Find the domain of

$$\frac{\sqrt{x - 18}}{\ln(y - 9) - 3}.$$

*Solution.* In the numerator, we have a square root function, which must have a nonnegative argument. So we need $x - 18 \geq 0$. In the denominator, we need $y - 9 > 0$, and we need $\ln(y - 9) - 3 \neq 0$. In other words, we need $x \geq 18$, $y > 9$ and $y \neq e^3 + 9$. In set notation,

$$\left\{(x, y) \colon x \geq 18,\, y > 9,\, y \neq e^3 + 9\right\}. \qquad \square$$

**Remark.** For quizzes it doesn't really matter the form you put these inequalities. In the previous example, we could have written

$$\left\{(x, y) \colon x - 18 \geq 0,\, y - 9 > 0,\, \ln(y - 9) - 3 \neq 0\right\}$$

and called it a day. But for the homework and exams these show up as multiple choice questions, so you should be comfortable manipulating these expressions to match them up with the correct answer choice.

The final idea we wish to introduce is level curves. In functions of two variables, we have that every output corresponds to an input of two values. This means that the graph of a function of two variables is 3-dimensional. These graphs get complicated pretty quickly and can be tough to visualize on paper. One way we deal with this is to use level curves. You've already seen a graph of level curves if you've ever looked at a topographical map. The idea is to look at various values of $z$ and see what these cross sections look like.

**Example 5.** If

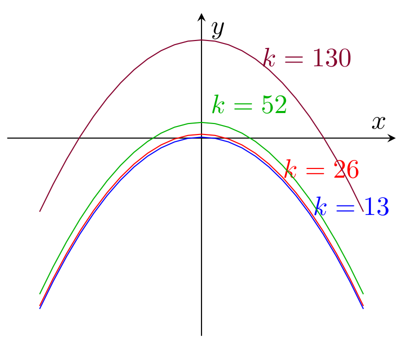
$$f(x,y) = 13\sqrt{y+7x^2},$$

then what do the level curves look like?

*Solution.* We consider $f(x,y) = k$, where $k$ is a constant.

$$\begin{aligned} k &= 13\sqrt{y+7x^2} \\ \frac{k}{13} &= \sqrt{y+7x^2} \\ \left(\frac{k}{13}\right)^2 &= y+7x^2 \\ y &= -7x^2 + \left(\frac{k}{13}\right)^2. \end{aligned}$$

Since $k$ is a constant, this is a parabola. If we wanted to graph the level curves for various values of $k$, it would look like




□

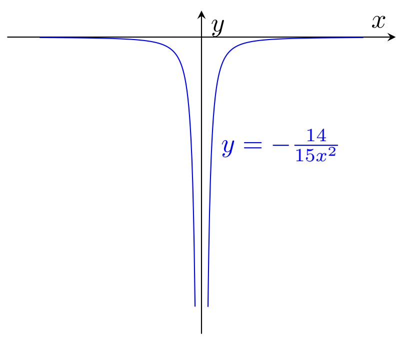
**Example 6.** Sketch the level curves for the function

$$f(x,y) = 15x^2y$$

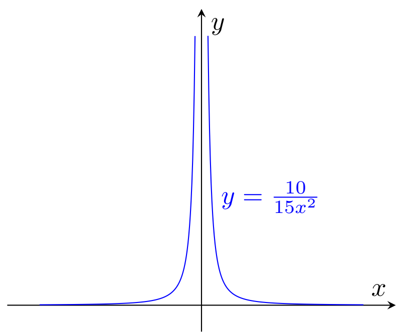
for $z = -14$, 10.

*Solution.* Setting $-14 = 15x^2y$, we solve for $y$ to get $y$ as a function of $x$

$$y = \frac{-14}{15x^2}.$$




And similarly for $z = 10$, we find $y = \frac{10}{15x^2}$.




$\square$

**Example 7.** Describe the level curves of the function

$$f(x, y) = \ln(x^2 + y^2).$$

*Solution.* Again we set $k = \ln(x^2 + y^2)$. We don't like the $x^2$ and $y^2$ being inside the logarithm, so we rewrite this as

$$e^k = x^2 + y^2.$$

Since $e^k$ is a constant, these are circles with center $(0, 0)$ and radius $e^{k/2}$. $\square$

It would be worthwhile to review equations of some basic curves that we may come across when studying level curves.

- ▷ $a(x - h)^2 + b(x - k)^2 = c$
  - Center at $(h, k)$
  - If $a \neq b$, this is an ellipse
  - If $a = b$, this is a circle with radius $\sqrt{\frac{c}{a}}$
- ▷ $a(x - h)^2 + b(y - h) = c$
  - Parabola

  - Vertex at $(h, \frac{c}{b} + k)$

- ▷ $a(x-h) + b(y-h)^2 = c$
  - (Sideways) parabola
  - Vertex at $(k, \frac{c}{a} - h)$

- ▷ $a(x-h) + b(y-k) = c$
  - Line
  - Slope $m = -\frac{a}{b}$

**Note.** In the homework there is a single word problem that deals with revenue. Recall that revenue is the amount of money you bring in. One way to formulate this is

$$R = pq,$$

where $p$ = price and $q$ = quantity. In a basic economics class one learns that quantity is a function of the price, and this is called the *demand*.