Overview

Being an applied calculus course, we love application problems. So this lesson contains no new information, and the homework is a variety of word problems using the second derivative test to find maxima and minima. Recall that the second derivative test will be given to you on your formula sheet.

Examples

Example 1. An airline places a restriction for the size carry-on luggage to a maximum of 68 inches for the length and girth combined. What is the maximum volume you can have in a carry-on given this constraint?

Solution. If the dimensions of the luggage are $l \times w \times h$, then the girth is $2w+2h$, assuming that l is the largest dimension. So our constraint is $68 = l + 2w + 2h$, and with that in mind, we want to maximize $V = lwh$. We can solve our constraint for l and use this to make V a function of just two variables:

$$\boxed{l = 68 - 2w - 2h,}$$

and

$$\begin{aligned} V &= (68 - 2w - 2h)wh \\ &= 68wh - 2w^2h - 2wh^2. \end{aligned}$$

Calculating the first partial derivatives,

$$V_w = 68h - 4wh - 2h^2$$

and

$$V_h = 68w - 2w^2 - 4wh.$$

Setting the first one equal to 0,

$$\begin{aligned} 68h - 4wh - 2h^2 &= 0 \\ h(68 - 4w - 2h) &= 0. \end{aligned}$$

Since we can't have $h = 0$, this tells us that

$$\boxed{h = 34 - 2w.}$$

Plugging this into V_h and setting that equal to 0,

$$\begin{aligned} 68w - 2w^2 - 4w(34 - 2w) &= 0 \\ 68w - 2w^2 - 136w + 8w^2 &= 0 \\ -68w + 6w^2 &= 0 \\ 2w(3w - 34) &= 0. \end{aligned}$$

So $\boxed{w = \frac{34}{3}.}$ Plugging this into our expression for h, we find

$$\boxed{h = 34 - 2 \cdot \frac{34}{3} = \frac{34}{3}.}$$

Finally, plugging both w and h into our expression for l,

$$\begin{aligned} l &= 68 - 2 \cdot \frac{34}{3} - 2 \cdot \frac{34}{3} \\ &= 68 - 4 \cdot \frac{34}{3} \\ &= \frac{68}{3}. \end{aligned}$$

So the maximum volume is $\left(\frac{68}{3}\right)\left(\frac{34}{3}\right)^2$. We should verify that this is indeed a maximum. To do this, we use the SDT. Calculating the second partial derivatives,

$$\begin{aligned} V_{ww} &= -4h \\ V_{hh} &= -4w \\ V_{wh} &= 68 - 4w - 4h. \end{aligned}$$

So

$$\begin{aligned} D &= (-4h)(-4w) - (68 - 4w - 4h)^2 \\ &= 16\left(\frac{34}{3}\right)^2 - \left(68 - 4 \cdot \frac{34}{3} - 4 \cdot \frac{34}{3}\right)^2 \\ &= \frac{4624}{3} > 0. \end{aligned}$$

And $V_{ww} = -4\left(\frac{34}{3}\right) < 0$, so we really do have a maximum. □

Example 2. The volume of a soup can is about 95π milliliters. What is the least amount of material required to make a can? (You may assume that a soup can is a perfect cylinder with uniform thickness.)

Solution. We don't actually need Calc III to figure this one out. We want to *minimize* surface area, that is we want to minimize

$$S = 2\pi r^2 + 2\pi r h,$$

given that

$$V = \pi r^2 h = 95\pi. \qquad (*)$$

We can solve for h in $(*)$ to obtain $h = \frac{95}{r^2}$. Making this substitution in our equation for surface area, we get

$$\begin{aligned} S &= 2\pi r^2 + 2\pi r\left(\frac{95}{r^2}\right) \\ &= 2\pi r^2 + \frac{190\pi}{r}. \end{aligned}$$

Now taking the derivative with respect to r,

$$S'(r) = 4\pi r - \frac{190\pi}{r^2}.$$

Setting this equal to 0,

$$\begin{aligned} 0 &= 4\pi r - \frac{190\pi}{r^2} \\ 0 &= 4\pi r^3 - 190\pi \\ 190\pi &= 4\pi r^3 \\ \frac{190}{4} &= r^3 \\ \left(\frac{190}{4}\right)^{1/3} &= r. \end{aligned}$$

Since $h = \frac{95}{r^2}$, this gives $h = 380^{1/3}$. This gives us a surface area of

$$\begin{aligned} S &= 2\pi\,(47.5)^{2/3} + 2\pi\,(47.5)^{1/3}\,(380)^{1/3} \\ &\approx 247. \end{aligned}$$

We should verify that this is a minimum with the SDT. But with the way we have set this problem up, we only need the one-variable SDT.

$$\begin{aligned} S''(r) &= 4\pi + \frac{380\pi}{r^3} \\ S''(47.5^{1/3}) &= 4\pi + \frac{380\pi}{47.5} > 0, \end{aligned}$$

which tells us that $r = 47.5^{1/3}$ is a minimum. □

Example 3. Because all the stores stopped selling boxes, you decide to make a rectangular box with a volume of 19 cubic feet. The material for the top and bottom costs 8 dollars per square foot, and the material for the 4 sides costs 3 dollars per square foot. To the nearest cent, what is the minimum cost for such a box?

Solution. We want to keep volume at a constant $V = 19 = lwh$. We can write cost as a function of all three dimensions:

$$\begin{aligned} C(l, w, h) &= 8 \cdot 2lw + 3 \cdot 2wh + 3 \cdot 2lh \\ &= 16lw + 6wh + 6lh. \end{aligned}$$

Solving the volume equation for h and substituting in the cost equation, we obtain a function of two variables

$$\begin{aligned} C(l, w) &= 16lw + 6w\left(\frac{19}{lw}\right) + 6l\left(\frac{19}{lw}\right) \\ &= 16lw + \frac{114}{l} + \frac{114}{w}. \end{aligned}$$

We want to *minimize* the cost function, so we take the first partial derivatives and set them equal to zero.

$$C_l = 16w - \frac{114}{l^2} = 0 \quad (1)$$

$$C_w = 16l - \frac{114}{w^2} = 0 \quad (2)$$

Solving for w in in (1), we obtain

$$16w = \frac{114}{l^2}$$

$$\boxed{w = \frac{7.125}{l^2}}$$

Making this substitution in (2),

$$\begin{aligned} 0 &= 16l - \frac{114}{(7.125/l^2)^2} \\ &= 16l - 114\left(\frac{l^2}{7.125}\right)^2 \\ &= 16l - \frac{16}{7.125}l^4 \\ &= 114l - 16l^4 \\ &= 16l(7.125 - l^3) \end{aligned}$$

Since $l = 0$ implies that $V = 0$ the only meaningful solution is $\boxed{l = \sqrt[3]{7.125}}$ and plugging this into what we found for w, we see that $\boxed{w = \sqrt[3]{7.125}}$. Note that this means that $h = 19/(7.125)^{2/3}$. Calculating the cost at this point,

$$\begin{aligned} C(\sqrt[3]{7.125}, \sqrt[3]{7.125}) &= 16\sqrt[3]{7.125^2} + 6\sqrt[3]{7.125}\left(\frac{19}{\sqrt[3]{7.125^2}}\right) + 6\sqrt[3]{7.125}\left(\frac{19}{\sqrt[3]{7.125^2}}\right) \\ &= 16\sqrt[3]{7.125^2} + \frac{228}{\sqrt[3]{7.125}} \\ &\approx 177.73. \end{aligned}$$

We should probably verify that this is indeed a minimum for the cost function. To do this, we use the second derivative test. Calculating the second partials,

$$\begin{aligned} C_{ll} &= \frac{228}{l^3} \\ C_{ww} &= \frac{228}{w^3} \\ C_{lw} &= 16. \end{aligned}$$

So

$$D = \frac{228}{7.125}\frac{228}{7.125} - 16^2 > 0,$$

and $C_{ll}(\sqrt[3]{7.125}, \sqrt[3]{7.125}) > 0$, so this really is a minimum. □

Example 4. A manufacturer is planning to sell a new product at the price of 400 dollars per unit and estimates that if x thousand dollars is spent on development and y thousand dollars is spent on promotion, consumers will buy approximately $\frac{110y}{y+5} + \frac{260x}{x+8}$ units of the product. If manufacturing costs for the product are 230 dollars per unit, how much should the manufacturer spend on development and how much on promotion to generate the largest possible profit?

Solution. Recall that profit is equal to revenue minus cost, and revenue is price times quantity:

$$P = R - C \qquad \text{and} \qquad R = pq.$$

We haven't talked about cost as much, but one way to describe cost is in terms of *variable cost* and *fixed cost*. As the names suggest, variable cost depends on the quantity, and fixed cost does not.

The revenue here is pretty easy to find, the price is \$400, and the quantity is the demand function. So,

$$R = 400\left(\frac{110y}{y+5} + \frac{260x}{x+8}\right).$$

For cost, we are told that we have a variable cost of \$230/unit. Our fixed costs are the amount we choose to spend on development and promotion, x and y respectively. But since x and y are in *thousands* of dollars, we need to multiply them each by 1000 to represent the actual money spent. Thus

$$C = 230\left(\frac{110y}{y+5} + \frac{260x}{x+8}\right) + 1000x + 1000y.$$

Putting this together, we have

$$\begin{aligned} P &= R - C \\ &= 400\left(\frac{110y}{y+5} + \frac{260x}{x+8}\right) - \left[230\left(\frac{110y}{y+5} + \frac{260x}{x+8}\right) + 1000x + 1000y\right] \\ &= 170\left(\frac{110y}{y+5} + \frac{260x}{x+8}\right) - 1000x - 1000y \\ &= \frac{18700y}{y+5} + \frac{44200x}{x+8} - 1000x - 1000y. \end{aligned}$$

To find how much money we should spend on each category, we want to maximize the profit function by setting $P_x = 0$ and $P_y = 0$. Finding P_x and P_y is straightforward:

$$\begin{aligned} P_x &= \frac{44200(x+8) - 44200x}{(x+8)^2} - 1000 \\ &= \frac{353600}{(x+8)^2} - 1000 \end{aligned}$$

and

$$\begin{aligned} P_y &= \frac{18700(y+5) - 18700y}{(y+5)^2} - 1000 \\ &= \frac{93500}{(y+5)^2} - 1000. \end{aligned}$$

Now

$$\begin{aligned} 0 &= \frac{353600}{(x+8)^2} - 1000 \\ 1000 &= \frac{353600}{(x+8)^2} \\ (x+8)^2 &= \frac{353600}{1000} \\ x+8 &= \sqrt{\frac{353600}{1000}} \\ x &= \sqrt{\frac{353600}{1000}} - 8 \\ x &\approx 10.804 \end{aligned}$$

Similarly,

$$\begin{aligned} 0 &= \frac{93500}{(y+5)^2} - 1000 \\ 1000 &= \frac{93500}{(y+5)^2} \\ (y+5)^2 &= \frac{93500}{1000} \\ y+5 &= \sqrt{\frac{93500}{1000}} \\ y &= \sqrt{\frac{93500}{1000}} - 5 \\ y &\approx 4.670. \end{aligned}$$

Again, since x and y are in thousands of dollars, we should interpret this as we should spend \$10804 on development and \$4670 on promotion.

Finally, to verify that this is a maximum, we compute

$$P_{xx} = \frac{-707200}{(x+8)^3} \qquad P_{xy} = 0.$$
$$P_{yy} = \frac{-187000}{(y+5)^3}$$

And it's quick to see that

$$D = \left(\frac{-707200}{(x+8)^3}\right)\left(\frac{-187000}{(y+5)^3}\right) > 0$$

and $P_{xx} < 0$, so we have found a maximum. □

Remark. Even though we verified each time that we had a maximum/minimum as desired, in practice this is cumbersome and unnecessary. But it is a good and thorough habit to check if you have the time.