Overview

In the past two lessons we were concerned with finding maxima and minima of multivariate functions. A common type of problem is maximizing/minimizing a function given some constraint, for example minimizing surface area given a particular volume. For these types of questions there is a more methodical approach with the method of Lagrange multipliers.

Lesson

The basic setup for using Lagrange multipliers is that we are given a function $f(x, y)$ subject to some constraint $g(x, y) = k$, and we want to maximize or minimize f with the given constraint. We'll first describe the method in full detail then see it in action with a few examples.

Method of Lagrange Multipliers. Suppose that $g_x(x, y)$ and $g_y(x, y)$ aren't both zero whenever $g(x, y) = k$. Then to find the maximum and minimum values of $f(x, y)$ subject to the constraint $g(x, y) = k$ (assuming that they exist), do the following.

1. Find all values x, y, and λ (λ is a real number) such that

$$\begin{aligned} f_x &= \lambda g_x \\ f_y &= \lambda g_y \\ g(x, y) &= k \end{aligned}$$

2. Evaluate f at every point (x, y) found in Step 1. The largest of these is the maximum value of f and the smallest is the minimum value.

Example 1. Find the maximum value of the function

$$f(x, y) = e^{8xy}$$

subject to the constraint $x^2 + y^2 = 100$. Assume both x and y are positive.

Solution. Here $g(x, y) = x^2 + y^2 = 100$. Using Lagrange multipliers, we have

$$\begin{aligned} f_x &= 8ye^{8xy} = \lambda 2x = \lambda g_x && (1) \\ f_y &= 8xe^{8xy} = \lambda 2y = \lambda g_y && (2) \\ g(x, y) &= x^2 + y^2 = 100 && (3) \end{aligned}$$

Solving (1) for λ, we get

$$\begin{aligned} \lambda 2x &= 8ye^{8xy} \\ \lambda &= \frac{4}{x} ye^{8xy}. \end{aligned}$$

Note that we are allowed to divide by x because $x > 0$ by assumption (in particular, $x \neq 0$). Plugging this into λ in (2),

$$\begin{aligned} 8xe^{8xy} &= \lambda 2y \\ 8xe^{8xy} &= \frac{4}{x} ye^{8xy} 2y \\ 8x &= \frac{8}{x} y^2 \\ x^2 &= y^2. \end{aligned}$$

Now we use (3):

$$\begin{aligned} x^2 + y^2 &= 100 \\ x^2 + x^2 &= 100 \\ 2x^2 &= 100 \\ x^2 &= 50. \end{aligned}$$

Since x and y are both positive (by assumption), $x^2 = y^2$ implies that $x = y$. Since we found that $x^2 = 50$, this means that $xy = 50$. Thus the maximum value of f is $e^{8\cdot 50} = e^{400}$. □

Example 2. Find the points at which the minimum values of $f(x, y) = x^2 e^{y^2}$ subject to the constraint $4y^2 + 2x = 10$ occur.

Solution. We start by setting $f_x = \lambda g_x$ and $f_y = \lambda g_y$.

$$f_x = 2xe^{y^2} = 2\lambda \tag{4}$$

$$f_y = 2yx^2 e^{y^2} = 8y\lambda \tag{5}$$

Solving for λ in (4), we get $\lambda = xe^{y^2}$. Using this in (5),

$$\begin{aligned} 2yx^2 \cancel{e^{y^2}} &= 8yx\cancel{e^{y^2}} && (e^{y^2} \text{ is never } 0) \\ 2\cancel{y}x^2 &= 8\cancel{y}x && (y > 0 \text{ by assumption}) \\ 0 &= 4x - x^2 \\ 0 &= x(4 - x), \end{aligned}$$

which gives us solutions of $x = 0, x = 4$. Since we are looking for where the minimum of f occurs, we can immediately see that it must be at $x = 0$. Since e^{y^2} is always positive, the smallest f can possibly be is 0, and $f(0, y) = 0$ for any y. Plugging this into $g(x, y) = 10$, for $x = 0$,

$$\begin{aligned} 4y^2 + 2 \cdot 0 &= 10 \\ y^2 &= \frac{5}{2} \\ y &= \pm\sqrt{\frac{5}{2}}. \end{aligned}$$

In case you're not convinced that we've already found where the minimum occurs, when $x = 4$, we must have

$$\begin{aligned} 4y^2 + 8 &= 10 \\ 4y^2 &= 2 \\ y^2 &= \frac{1}{2} \\ y &= \pm\frac{1}{\sqrt{2}} \end{aligned}$$

But $f(4, \pm 1/\sqrt{2}) > f(0, \pm\sqrt{5/2})$. Thus the minimum value occurs at $(0, \sqrt{5/2})$ and $(0, -\sqrt{5/2})$. □

Example 3. Find the maximum of $f(x, y) = \ln(9xy^2)$ subject to the constraint $3x^2 + 8y^2 = 4$.

Solution. First we compute f_x and f_y and set them equal to λg_x and λg_y, respectively. Here $g(x, y) = 3x^2 + 8y^2$.

$$f_x = \frac{9y^2}{9xy^2} = \frac{1}{x} = 6\lambda x = \lambda g_x \tag{6}$$

$$f_y = \frac{18xy}{9xy^2} = \frac{2}{y} = 16\lambda y = \lambda g_y. \tag{7}$$

We can solve (6) for λ by dividing both sides by x. Note that this is allowed since $x \neq 0$ as $x = 0$ would give $f(0, y) = \ln 0$. So $\lambda = \frac{1}{6x^2}$. Plugging this into (7),

$$\begin{aligned} \frac{2}{y} &= 16\lambda y \\ \frac{2}{y} &= 16\left(\frac{1}{6x^2}\right)y \\ 12x^2 &= 16y^2 \\ x^2 &= \frac{16}{12}y^2 \\ x^2 &= \frac{4}{3}y^2. \end{aligned} \tag{8}$$

We can plug this into our constraint equation to get

$$\begin{aligned}
3x^2 + 8y^2 &= 4 \\
3\left(\frac{4}{3}y^2\right) + 8y^2 &= 4 \\
4y^2 + 8y^2 &= 4 \\
12y^2 &= 4 \\
y^2 &= \frac{1}{3} \\
y &= \pm\frac{1}{\sqrt{3}}.
\end{aligned}$$

Now using this in (8),

$$\begin{aligned}
x^2 &= \frac{4}{3}\cdot\frac{1}{3} \\
x^2 &= \frac{4}{9} \\
x &= \pm\frac{2}{3}.
\end{aligned}$$

But $x = -2/3$ is not possible since that would force us to take ln of a negative number. Thus our solutions are $(2/3, 1/\sqrt{3})$ and $(2/3, -1/\sqrt{3})$. Notice that $f(2/3, 1/\sqrt{3}) = f(2/3, -1/\sqrt{3})$ since we're squaring y. So we have a maximum of $\ln\left(9\cdot\frac{2}{3}\cdot\frac{1}{3}\right) = \ln 2$. □

Example 4. Find the minimum value of $f(x, y) = x^2 + y^2$ subject to the constraint $5y = 5 - 2x$.

Solution. Recall that in the method of Lagrange multipliers we need our constraint to be of the form $g(x, y) = k$. But this is no problem here since adding $2x$ to both sides of our constraint equation gives us $g(x, y) = 2x + 5y = 5$.

Setting $f_x = \lambda g_x$ and $f_y = \lambda g_y$,

$$f_x = 2x = 2\lambda = \lambda g_x \tag{9}$$
$$f_y = 2y = 5\lambda = \lambda g_y. \tag{10}$$

Right away in (9), we can see that $\lambda = x$. Plugging this into (10), we get

$$\begin{aligned}
2y &= 5\lambda \\
2y &= 5x \\
y &= \frac{5}{2}x.
\end{aligned}$$

Plugging this into our constraint,

$$\begin{aligned} 2x + 5y &= 5 \\ 2x + 5\left(\frac{5}{2}x\right) &= 5 \\ \frac{4}{2}x + \frac{25}{2}x &= 5 \\ \frac{29}{2}x &= 5 \\ x &= \frac{10}{29}. \end{aligned}$$

Plugging this into our equation for y, we find $y = \frac{25}{29}$. Thus our minimum is

$$\left(\frac{10}{29}\right)^2 + \left(\frac{25}{29}\right)^2 = \frac{25}{29}. \qquad \square$$