Overview

In this section we introduce integration by parts. So far our methods of integration include just knowing an anti-derivative by memory and u-substitution. As we add this new method of integration to our bag of tricks, we should still keep in mind the first two, and try them first, since they are generally less complicated.

Lesson

We don't have to look very hard before integrals become deceivingly difficult. Take for example

$$\int \ln x \, dx.$$

The tempting solution is to say $\frac{1}{x} + C$, but remember that $\frac{1}{x}$ is the *derivative* of $\ln x$. To answer this question, we need integration by parts. It's actually not too bad to derive the formula on our own; it requires little more than knowledge of the product rule for derivatives.

Consider $u(x)$, $v(x)$, two functions of x. Then the product rule tells us

$$\frac{d}{dx} u(x)v(x) = u(x)v'(x) + v(x)u'(x).$$

Integrating both sides with respect to x, we get

$$\begin{aligned} \int \frac{d}{dx} u(x)v(x) \, dx &= \int \Big[u(x)v'(x) \, dx + v(x)u'(x) \Big] \, dx \\ &= \int u(x)v'(x) \, dx + \int v(x)u'(x) \, dx. \end{aligned}$$

Recalling the fundamental theorem of calculus from first semester calculus, we see that the left hand side of this equation is just $u(x)v(x)$. To resolve the right hand side, we just need to know that by definition, $du = u'(x) \, dx$ and $dv = v'(x) \, dx$. Putting these facts together, we get

$$u(x)v(x) = \int u(x) \, dv + \int v(x) \, du.$$

Now solving for $\int u(x) \, dv$ and then dropping the x's to compactify notation, we get the integration by parts formula.

$$\boxed{\int u \, dv = uv - \int v \, du}$$

Now we should know how to find the anti-derivative for $\ln x$.

Example 1. Compute

$$\int \ln x \, dx.$$

Solution. We pick

$$u = \ln x \qquad dv = dx$$

$$du = \frac{1}{x} \, dx \qquad v = x$$

Then

$$\int \ln x\,dx = \underbrace{x\ln x}_{uv} - \underbrace{\int x\cdot\frac{1}{x}\,dx}_{\int v du}$$
$$= x\ln x - \int dx$$
$$= x\ln x - x + C$$

□

How do we know what to choose for u and dv? Well, our choice for u should be something we know the derivative of, and we should know the anti-derivative for dv. A useful guideline is the acronym "LATE." In general, we should try picking our u from this list, in the order which they appear. whatever isn't u should be dv.

- L: logarithmic
- A: algebraic (polynomials)
- T: trig
- E: exponential

After we do integration by parts, it may require a second, a third, or even more iterations before we reach a final answer. It is important to work just one step at a time and be sure to use parentheses when necessary to avoid making sign mistakes. It's good mathematical practice to use different letters when making new substitutions, but for your own scratch work, this is not absolutely essential.

Example 2. Compute

$$I = \int x^2\cos(5x)\,dx$$

Solution. We start by picking u and dv according to our guidelines.

$$u = x^2 \qquad dv = \cos(5x)\,dx$$
$$du = 2x \qquad v = \frac{1}{5}\sin(5x)$$

Then we will immediately see that the same type of problem appears in our expression for $\int v\,du$. We just keep repeating the process.

$$s = x \quad dt = \sin(5x)\,dx$$
$$ds = dx \quad t = -\frac{1}{5}\cos(5x)$$

$$I = \frac{x^2}{5}\sin(5x) - \int \frac{2}{5}x\sin(5x)\,dx$$
$$= \frac{x^2}{5}\sin(5x) - \frac{2}{5}\left[-\frac{x}{5}\cos(5x) + \frac{1}{5}\int\cos(5x)\,dx\right]$$
$$= \frac{x^2}{5}\sin(5x) - \frac{2}{5}\left[-\frac{x}{5}\cos(5x) + \frac{1}{5}\left(\frac{1}{5}\sin(5x)\right)\right]$$
$$= \frac{x^2}{5}\sin(5x) + \frac{2x}{25}\cos(5x) - \frac{2}{125}\sin(5x)$$

□

As we have started to delve into a new section it is easy to have ourselves only consider the new method for solving problems. It turns out this isn't always the best practice. As we've seen in the previous example, choosing $u = x^2$ made us do integration by parts twice. Try the following problem using parts, and you should find it requires three iterations, which is no fun. It turns out using a two-step u-substitution, we can solve this problem fairly easily.

Example 3. Compute

$$I = \int \frac{9x^3}{\sqrt{6+x^2}}\,dx$$

Solution. Let $u = x^2$, then $du = 2x\,dx$, so that $\frac{1}{2}\,du = dx$. Now this should start to look like a problem we've seen in previous lessons.

$$\begin{aligned} I &= 9\int \frac{x^2}{\sqrt{6+x^2}}\,x\,dx \\ &= \frac{9}{2}\int \frac{u}{\sqrt{6+u}}\,du && \begin{aligned} u &= x^2 \\ du &= 2x\,dx \end{aligned} \\ &= \frac{9}{2}\int \frac{s-6}{\sqrt{s}}\,ds && \begin{aligned} s &= 6+u \\ ds &= du \end{aligned} \\ &= \frac{9}{2}\int \left(s^{1/2} - 6s^{-1/2}\right)\,ds \\ &= \frac{9}{2}\left(\frac{2}{3}s^{3/2} - 12s^{1/2}\right) + C \\ &= 3s^{3/2} - 54s^{1/2} + C \\ &= 3(6+u)^{3/2} - 54(6+u)^{1/2} + C \\ &= 3(6+x^2)^{3/2} - 54(6+x^2)^{1/2} + C. \end{aligned}$$

□

Example 4. Compute

$$I = \int (t+13)e^{20-t}\,dt$$

Solution. There is nothing special about this example. We choose our u and dv according to our guidelines, and should arrive at the solution quickly.

$$\begin{aligned} u &= t + 13 & dv &= e^{20-t}\,dt \\ du &= dt & v &= -e^{20-t} \end{aligned}$$

$$\begin{aligned} I &= (t+13)(-e^{20-t}) - \int -e^{20-t}\,dt \\ &= -(t+13)e^{20-t} - e^{20-t} + C \end{aligned}$$

□

The next example is only slightly more complicated, requiring a second iteration of integration by parts.

Example 5. Compute

$$I = \int (5z^2+3)e^{10z}\,dz$$

Solution. We pick our u and dv :

$$u = 5z^2 + 3 \qquad dv = e^{10z}\, dz$$
$$du = 10z \qquad v = \frac{1}{10}e^{10z}\, dz$$

$$\begin{aligned} I &= \frac{1}{10}(5z^2+3)e^{10z} - \int 10z(\frac{1}{10}e^{10z})\, dz \\ &= \frac{1}{10}(5z^2+3)e^{10z} - \int ze^{10z}\, dz \\ &= \frac{1}{10}(5z^2+3)e^{10z} - \left(\frac{z}{10}e^{10z} - \int \frac{1}{10}e^{10z}\, dz\right) \\ &= \frac{1}{10}(5z^2+3)e^{10z} - \frac{z}{10}e^{10z} + \frac{1}{100}e^{10z} + C \\ &= \frac{1}{100}e^{10z}\left(50z^2 - 10z + 31\right) + C \end{aligned}$$

$$s = z \quad dv = e^{10z}\, dz$$
$$ds = dz \quad v = \frac{1}{10}e^{10z}$$

□

Note. As always don't forget your "$+\,C$"!