Overview

In this lesson we expose ourselves to more examples of integration by parts, as well as introduce definite integrals which require integration by parts. We also add to our repertoire of styles of word problems we should be able to figure out. Let's recall our all-important formula for this lesson:

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Lesson

We begin with a warmup.

Warmup. Find the are of the region under the curve

$$f(x) = x(x-3)^2, \quad 3 \le x \le 6.$$

Solution. There are actually at least 3 ways to come to a solution to this problem: we could multiply out $(x-3)^2$ and then distribute the x. Then we would just have a power rule problem. We could also use u-substitution, picking $u = x - 3$. This would yield $\int_0^3 (u+3)u^2\,du$, and then we could distribute the u^2 term to get another power rule problem. Both of these are probably preferable to integration by parts for this problem, as we will see it's a bit more complicated than we need. To start, we pick our u and dv:

$$\begin{aligned} u &= x & dv &= (x-3)^2 \\ du &= dx & v &= \frac{1}{3}(x-3)^3 \end{aligned}$$

We'll see that this is a good choice since the x disappears in the next integral.

$$\begin{aligned} A &= \left[\frac{1}{3}x(x-3)^3 - \frac{1}{3}\int (x-3)^3\,dx\right]_3^6 \\ &= \left(\frac{1}{3}x(x-3)^3 - \frac{1}{12}(x-3)^4\right)\Bigg|_3^6 \\ &= \frac{189}{4} \end{aligned}$$

□

Remark. Even though during integration by parts we have u's and v's flying around, it is not actually a u-substitution. This means that we don't have to change the bounds of integration since our integral is still in terms of x.

Interlude on definite integrals

It is important for us to recognize what it means to compute a definite integral in various applied contexts. For example, if $v(t)$ represents a velocity, then

$\int_a^b v(t)\,dt$ represents the *displacement*, or *net distance traveled*. What we mean by this is that if we were to take $\int_a^b |v(t)|\,dt$ we would get the *total distance traveled*. Since in general a velocity could be negative, it's preferable to say the integral of velocity gives us displacement. In the context of a given problem, if we know velocity is always positive on the specified interval, we need not make this distinction.

Velocity is the prime example, but it should be easy to extend this idea to other contexts. For example if $r(t)$ represents the rate of production of something, $\int_a^b r(t)\,dt$ would give you the (net) production of that thing. If $r(t)$ represents a rate of growth, then $\int_a^b r(t)\,dt$ represents the net growth on the interval $a \leq t \leq b$. Let's look at some examples that illustrate this.

Example 1. An ant is traveling along a picnic table at a velocity of

$$v(t) = 120te^{-2t} \quad \text{mm/s}, \quad 0 \leq t \leq 60.$$

Assuming the ant was initially at rest, what is the distance traveled during those 60 seconds?

Solution. From the discussion above, it should be straightforward to see that we want to integrate $v(t)$ on the interval $0 \leq t \leq 60$. So the distance during those 60 seconds is given by

$$D = \int_0^{60} 120te^{-2t}\,dt.$$

$$u = t \qquad dv = e^{-2t}\,dt$$

$$du = dt \qquad v = -\frac{1}{2}e^{-2t}$$

We should see pretty quickly that this is going to require integration by parts, picking u and dv as follows.

$$\begin{aligned} D &= 120\left(-\frac{1}{2}te^{-2t} + \frac{1}{2}\int e^{-2t}\,dt\right) \\ &= 120\left[-\frac{1}{2}te^{-2t} - \frac{1}{4}e^{-2t}\right]_0^{60} \\ &\approx 30 \text{ mm.} \end{aligned}$$

□

The next example is a word problem we've encountered before, but in light of what we've been doing in this lesson, there is another way to think about it.

Example 2. A baby is born at a height of 1.5 feet and during the first 7 years, he grows at a rate of

$$r(t) = \frac{30\ln\sqrt{t+6}}{(t+6)^2} \quad \text{feet/year.}$$

How tall is he at 7 years old?

Solution. Let $h(t)$ be the height at time t. Before, we may have only thought to find the antiderivative of $r(t)$, figure out what C is, then evaluate $h(7)$. Another way to think about it is to recognize that if we integrate $\int_0^7 r(t)\,dt$, this will give us the change in height during the first 7 years. Since we know the initial height $h(0) = 1.5$, we can add $h(0)$ to $\int_0^7 r(t)\,dt$ to get $h(7)$. So,

$$\begin{aligned}
h(7) &= 1.5 + \int_0^7 \frac{30 \ln \sqrt{t+6}}{(t+6)^2}\,dt \\
&= 1.5 + 30 \int_0^7 \ln \sqrt{t+6}(t+6)^{-2}\,dt \\
&\overset{(*)}{=} 1.5 + 30 \left[-\ln \sqrt{t+6}\frac{1}{t+6} + \frac{1}{2}\int \frac{1}{(t+6)^2}\,dt \right] \\
&= 1.5 + 30 \left[-\ln \sqrt{t+6}\frac{1}{t+6} - \frac{1}{2(t+6)} \right]_0^7 \\
&\approx 1.5 + 2.866 \\
&= 4.366\,\text{feet}.
\end{aligned}$$

To get $(*)$, we used integration by parts with:

$$\begin{aligned}
u &= \ln \sqrt{t+6} & dv &= (t+6)^{-2}\,dt \\
du &= \frac{1}{2(t+6)} & v &= -\frac{1}{t+6}
\end{aligned}$$

□

Example 3. Find the area bounded by the curves

$$y = 9x^3 \ln x,\ y = 0,\ x = 1,\ x = 7.$$

Solution. Setting up integrals for areas bounded by curves should come fairly naturally at this point. What may still be a little illusive is our choice of u and dv. As I've mentioned in class, polynomials are a great choice for u, but so is $\ln x$. But $\ln x$ takes precedence for choice of u over polynomials.

$$\begin{aligned}
u &= \ln x & dv &= x^3\,dx \\
du &= \frac{1}{x} & v &= \frac{1}{4}x^4
\end{aligned}$$

$$\begin{aligned}
A &= 9\int_1^7 x^3 \ln x\,dx \\
&= 9\left(\frac{1}{4}x^4 \ln x - \frac{1}{4}\int \frac{1}{x}x^4\,dx\right)\Bigg|_1^7 \\
&= 9\left(\frac{1}{4}x^4 \ln x - \frac{1}{4}\int x^3\,dx\right)\Bigg|_1^7 \\
&= 9\left(\frac{1}{4}x^4 \ln x - \frac{1}{16}x^4\right)\Bigg|_1^7 \\
&= 9 \cdot 7^4 \left[\frac{1}{4}\ln 7 - \frac{1}{16}\right] - \frac{9}{16}
\end{aligned}$$

□

The next example definitely has a different feel from the previous word problems. Here we are given a continuous function that describes a probability of a particular situation. This is called a probability density function, and the particular situation is called a random variable. If you were to take a course in probability, you would learn that if $f_X(x)$ is a probability density function for a random variable X, then the probability that $a \leq X \leq b$ is given by

$$P[a \leq X \leq b] = \int_a^b f_X(x)\, dx.$$

This may seem overly complicated, and if the previous paragraph just sounds like a bunch of abstract nonsense, that's okay. We do need to know how to put this into action though.

Example 4. During a crazy weekend on a college campus, random samples of students are given a field sobriety test. The probability of finding a sample that has x percentage of students passing is described by

$$\frac{2x}{\sqrt{1+8x}}, \tag{$*$}$$

where x is a number between 0 and 1. What is the probability that a tested sample of students has at least 75% passing the sobriety test?

Solution. In this example, our random variable X is the percentage of students passing the sobriety test, given as a decimal between 0 and 1. Moreover, the function in $(*)$ is our probability density function. If we want at least 75% of the students to pass, this translates to $X \geq .75$. But since the percentage can be no more than 100%, this tells us that we are looking for $P[.75 \leq x \leq 1]$. So,

$$\begin{aligned}
P[\geq 75\% \text{ pass}] &= \int_{.75}^1 \frac{2x}{\sqrt{1+8x}}\, dx \\
&= 2\int_{.75}^1 \frac{x}{\sqrt{1+8x}}\, dx \qquad \begin{array}{ll} u = x & dv = (1+8x)^{-1/2} \\ du = dx & v = \frac{2}{8}(1+8x)^{1/2} \end{array} \\
&= 2\left[\frac{1}{4}x(1+8x)^{1/2} - \int \frac{1}{4}(1+8x)^{1/2}\, dx\right]_{.75}^1 \\
&= \frac{1}{2}\left[x(1+8x)^{1/2} - \frac{1}{12}(1+8x)^{3/2}\right]_{.75}^1 \\
&\approx .375 - .220479 \\
&= .154521
\end{aligned}$$

Thus there is a 15.45% chance that we would pick a random sample of students with at least 75% of them passing the sobriety test. □

Remark. We could have done this problem with regular u-substitution. For let

$$\begin{aligned} u &= 1 + 8x \\ du &= 8\,dx. \end{aligned}$$

Then the integral becomes

$$\begin{aligned} \frac{1}{8}\int_{x=.75}^{x=1} \frac{\frac{1}{4}(u-1)}{u^{1/2}}\,du &= \frac{1}{32}\int_7^9 \frac{u-1}{u^{1/2}}\,du \\ &= \frac{1}{32}\int_7^9 \left(u^{1/2} - u^{-1/2}\right)\,du \\ &= \frac{1}{32}\left[\frac{2}{3}u^{3/2} - 2u^{1/2}\right]_7^9 \\ &= \frac{1}{32}\left[\left(\frac{2}{3}(9)^{3/2} - 2(9)^{1/2}\right) - \left(\frac{2}{3}(7)^{3/2} - 2(7)^{1/2}\right)\right] \\ &\approx .154521. \end{aligned}$$