There is no new information in this lesson. We continue working on separable equations, adding more examples.

Examples

Example 1. Solve the initial value problem

$$\frac{dy}{dt} + t^k y = 0, \quad y(0) = 1, \quad y(1) = e^{-11}.$$

Solution. If this were not an initial value problem, we would need to separately consider the case $k = -1$ from the case $k \neq -1$. But if you were to consider the differential equation $\frac{dy}{dt} + t^{-1}y = 0$ with the same initial conditions, you would find that we would have to try to take $\ln 0$, which is undefined.

With that out of the way, subtracting $t^k y$ from both sides,

$$\begin{aligned} \frac{dy}{dt} &= -t^k y \\ \frac{dy}{y} &= -t^k\, dt \\ \int \frac{dy}{y} &= \int -t^k\, dt \\ \ln|y| &= \frac{-t^{k+1}}{k+1} + C \qquad (1) \end{aligned}$$

Using $y(0) = 1$,

$$\begin{aligned} \ln 1 &= \frac{0^{k+1}}{k+1} + C \\ 0 &= 0 + C. \end{aligned}$$

So $C = 0$, and using $y(1) = e^{-11}$,

$$\begin{aligned} \ln e^{-11} &= \frac{-1^{k+1}}{k+1} \\ -11 &= \frac{-1}{k+1} \\ k+1 &= \frac{-1}{-11} \\ k+1 &= \frac{1}{11}. \end{aligned}$$

Putting this back into (1), we get

$$\begin{aligned} \ln|y| &= \frac{-t^{1/11}}{1/11} \\ \ln|y| &= -11t^{1/11} \\ y &= e^{-11t^{1/11}}. \end{aligned}$$

□

Example 2. Solve the initial value problem

$$\frac{dy}{dt} + y\sin t = 0, \quad y(\pi) = -7.$$

Solution. Again we start by subtracting the non-$\frac{dy}{dt}$ term from both sides.

$$\begin{aligned}
\frac{dy}{dt} &= -y\sin t \\
\frac{dy}{y} &= -\sin t\, dt \\
\int \frac{dy}{y} &= \int -\sin t\, dt \\
\ln|y| &= \cos t + C_1 \\
y &= e^{\cos t + C_1} \\
y &= e^{C_1} e^{\cos t} \\
y &= Ce^{\cos t} \qquad (2)
\end{aligned}$$

Using $y(\pi) = -7$,

$$\begin{aligned}
-7 &= Ce^{\cos \pi} \\
-7 &= Ce^{-1} \\
C &= -7e
\end{aligned}$$

Putting this back into (2),

$$\begin{aligned}
y &= -7e \cdot e^{\cos t} \\
y &= -7e^{1+\cos t}
\end{aligned}$$

□

Example 3. Find a general solution to the differential equation

$$\frac{dy}{dt} + 13y = 0.$$

Solution.

$$\begin{aligned}
\frac{dy}{dt} &= -13y \\
\frac{dy}{y} &= -13\, dt \\
\int \frac{dy}{y} &= \int -13\, dt \\
\ln|y| &= -13t + C \\
|y| &= e^{-13t + C_1} \\
y &= \pm e^{C_1} e^{-13t} \\
y &= Ce^{-13t}.
\end{aligned}$$

Note that since $\pm e^{C_1}$ is just some constant, we can get rid of the $\pm$ and just call this our new C. $\square$

Example 4. Mumps is spreading on a college campus at a rate proportional to the infected population. On the day of the outbreak there are 5 people infected. A week later, there are 8 people infected. If there is no intervention, how many people will have been infected 30 days from the outbreak?

Solution. Let $P(t)$ represent the number of people that have been infected at time t days from the outbreak. Then $P(0) = 5$, $P(7) = 8$, and we wish to know $P(30)$. The usual differential equation gives us

$$\frac{dP}{dt} = kP.$$

As we have seen several times, this has as a solution

$$P(t) = Ce^{kt},$$

and it is easy to see that $C = P(0) = 5$. Solving for k,

$$\begin{aligned} P(7) = 8 &= 5e^{k\cdot 7} \\ \frac{8}{5} &= e^{k\cdot 7} \\ \ln\frac{8}{5} &= k\cdot 7 \\ \frac{1}{7}\ln\frac{8}{5} &= k \\ .06714 &\approx k \end{aligned}$$

Now computing $P(30)$,

$$\begin{aligned} P(30) &= 5e^{.06714\cdot 30} \\ &\approx 37.47. \end{aligned}$$

So we would expect about 37 people to become infected. $\square$

Example 5. You decide to hang-dry your clothes to save money. They dry out at a rate proportional to the moisture content. If after 1 hour they have lost 15% of their moisture, how long will it take for your clothes to lose 90% of their moisture?

Solution. We'll call 100% of the clothing's moisture content the amount of moisture there is at time $t = 0$. Let $y(t)$ denote the percentage of moisture *remaining* in the clothing at time t hours since pulling them out of the wash. Then we have

$$y = e^{kt}$$

since this is the same differential equation from the previous example, just with $y(0) = 1$. The other piece of information we have is $y(1) = 1 - .15 = .85$. We use this to solve for k

$$\begin{aligned} .85 &= e^k \\ \ln .85 &= k. \end{aligned}$$

So the equation is

$$\begin{aligned} y &= e^{t \ln .85} \\ &= e^{\ln[(.85)^t]} \\ &= (.85)^t. \end{aligned}$$

Note that we didn't have to do those last few steps, but don't fall into the trap of 'cancelling' out the e and the ln. Always follow the real laws of logarithms and exponentials!

Back to the problem... we're looking for what t is there .1 of the moisture content remaining. So

$$\begin{aligned} .1 &= e^{t \ln .85} \\ \ln .1 &= t \ln .85 \\ \frac{\ln .1}{\ln .85} &= t \\ 15 &\approx t. \end{aligned}$$

□