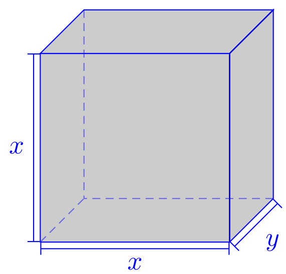
**Instructions.** Show all work, with clear logical steps. No work or hard-to-follow work will lose points.

**Problem 1.** (10 points) A rectangular building with a square front is to be constructed of materials that costs 19 dollars per $\text{ft}^2$ for the flat roof, 11 dollars per $\text{ft}^2$ for the sides and the back, and 13 dollars per $\text{ft}^2$ for the glass front. There are literally no other costs associated with constructing this fictitious building. If the volume of the building is 5,600 $\text{ft}^3$, what dimensions will minimize the cost of materials?

*Solution.* Let's draw a picture to represent the building described.




Then our constraint is $V(x, y) = x^2 y = 5600$, and the function which we want to minimize is cost. The cost is given by

$$\begin{aligned} C(x,y) &= \underbrace{19xy}_{\text{top}} + \underbrace{2 \cdot 11xy}_{\text{left/right}} + \underbrace{11x^2}_{\text{back}} + \underbrace{13x^2}_{\text{front}} \\ &= 41xy + 24x^2. \end{aligned}$$

Using the method of Lagrange multipliers,

$$C_x = 41y + 48x = \lambda 2xy = \lambda V_x \tag{1}$$

$$C_y = 41x = \lambda x^2 = \lambda V_y. \tag{2}$$

Using (2), we have

$$\begin{aligned} 0 &= \lambda x^2 - 41x \\ &= x(\lambda x - 41), \end{aligned}$$

which gives either $x = 0$ or $\lambda x = 41$. We can't have $x = 0$ because this would give us $V = 0$. Using $\lambda x = 41$ in (1),

$$\begin{aligned} 41y + 48x &= 2(\lambda x)y \\ 41y + 48x &= 2(41)y \\ 48x &= 41y \\ \end{aligned}$$

$$\boxed{x = \frac{41}{48}y.}$$

Using this in our constraint equation, we get

$$\begin{aligned}
x^2y &= 5600 \\
\left(\frac{41}{48}y\right)^2 y &= 5600 \\
y^3 &= 5600\left(\frac{48}{41}\right)^2 \\
y &= \sqrt[3]{5600\left(\frac{48}{41}\right)^2} \\
y &\approx 19.7258.
\end{aligned}$$

Finally, using the boxed equation for $x$ above, we find

$$x \approx \frac{41}{48}(19.7258) \approx 16.8491. \qquad \square$$

**Remark.** You didn't have to do this with Lagrange multipliers. This problem is actually doable with one-variable calculus. We can solve the constraint equation for $y$. Note that $x \neq 0$ since we have $x^2y = 5600$. This gives us

$$y = \frac{5600}{x^2}.$$

Plugging this into our cost function, we get

$$\begin{aligned}
C(x) &= 41x\left(\frac{5600}{x^2}\right) + 24x^2 \\
&= \frac{229600}{x} + 24x^2.
\end{aligned}$$

If we calculate $C'(x)$ and set this equal to 0, we will find the $x$ at which the minimum occurs, and then we can use this to solve for $y$.