Instructions. Show all work, with clear logical steps. No work or hard-to-follow work will lose points.

Problem 1. Solve the initial value problem

$$y' = kt^n, \qquad y(0) = 7.$$

where k, n are constants and $n \neq -1$. [Notes: your answer will have both k's and n's in it.]

Solution. We have $dy = kt^n\,dt$. Since $n \neq -1$, this is just the power rule, and

$$\begin{aligned} \int dy &= \int kt^n\,dt \\ y &= \frac{k}{n+1}t^{n+1} + C. \end{aligned}$$

Since $y(0) = 7$, we have $C = 7$. This gives us

$$y = \frac{k}{n+1}t^{n+1} + 7. \qquad \square$$

Problem 2. Say you brew coffee in your cozy 22°C dorm room at 100°C. After 5 minutes, it cools to 95°C. How much longer will it take to cool to 85°C? Approximate to two decimal places.

Solution. Recall Newton's Law of Cooling:

$$\frac{dT}{dt} = k(T - T_A).$$

Here $T_A = 22,\ T(0) = 100,\ T(5) = 95$. Then

$$\begin{aligned} \frac{dT}{dt} &= k(T - 22) \\ \frac{dT}{T - 22} &= k\,dt \\ \int \frac{dT}{T - 22} &= \int k\,dt \\ \ln(T - 22) &= kt + C. \end{aligned}$$

Note that we don't need $\ln|T - 22|$ since the temperature of the coffee can't drop below 22. Solving for C, we use $T(0) = 100$.

$$\begin{aligned} \ln(100 - 22) &= k \cdot 0 + C \\ \ln 78 &= C. \end{aligned}$$

Now we use $T(5) = 95$ to find k

$$\begin{aligned}
\ln(95-22) &= kt + \underbrace{\ln 78}_{C} \\
\ln 73 &= k \cdot 5 + \ln 78 \\
\ln 73 - \ln 78 &= k \cdot 5 \\
\ln \frac{73}{78} &= k \cdot 5 \\
\frac{1}{5} \ln \frac{73}{78} &= k.
\end{aligned}$$

Solving for T, we have

$$\begin{aligned}
\ln(T-22) &= \frac{1}{5} t \ln \frac{73}{78} + \ln 78 \\
T - 22 &= e^{\frac{1}{5} t \ln \frac{73}{78} + \ln 78} \\
T &= 22 + e^{\frac{1}{5} t \ln \frac{73}{78}} e^{\ln 78} \\
&\boxed{T = 22 + 78 e^{\frac{1}{5} t \ln \frac{73}{78}}}
\end{aligned}$$

But what we're really after is how long until $T = 85$.

$$\begin{aligned}
\ln(85-22) &= \frac{1}{5} t \ln \frac{73}{78} + \ln 78 \\
\ln 63 - \ln 78 &= \frac{1}{5} t \ln \frac{73}{78} \\
5 \ln \frac{63}{78} &= t \ln \frac{73}{78} \\
\frac{5 \ln \frac{63}{78}}{\ln \frac{73}{78}} &= t \\
&\boxed{t \approx 16.11}
\end{aligned}$$

So when $t = 16.11$ the coffee has cooled to the desired temperature, which means in $16.11 - 5 = 11.11$ more minutes. □