Overview

In this lesson we discuss higher order partial derivatives. If you know how to compute a partial derivative, you know how to compute any number of them.

Lesson

Definition. Given a function $z = f(x, y)$, the *partial derivative with respect to x* is the (one-variable) derivative with respect to x, holding y constant. Of course we have the respective statement for the partial derivative with respect to y.

As with any new idea, we need to introduce some notation for partial derivatives. We describe what we do with x.

Notation. We have several ways we could write "the partial derivative of $z = f(x, y)$ with respect to x:"

$$\frac{\partial z}{\partial x}, \quad \frac{\partial}{\partial x} f(x, y), \quad f_x, \quad \frac{\partial f}{\partial x}$$

If we want to evaluate the partial derivative at a point (x_0, y_0), we could write:

$$\left.\frac{\partial z}{\partial x}\right|_{(x_0, y_0)}, \quad f_x(x_0, y_0)$$

We will most commonly use the f_x, f_y notation for partial derivatives, but the others do appear from time to time, so it is important to be comfortable with all the notations. As far as computation goes, there is nothing more to partial derivatives than their definition, so we illustrate with some examples.

Example 1. Given

$$f(x, y) = e^{10x^2+9y^2},$$

find f_x and f_y.

Solution. We start by taking f_x. Keep in mind that any instance of y is a constant, so we can just think of y as our favorite number.

$$f_x = 20xe^{10x^2+9y^2}$$

We have $e^{10x^2+9y^2}$ since we are differentiating an exponential, the $20x$ comes from the chain rule, and $\frac{\partial}{\partial x}(9y^2) = 0$. The same reasoning gives us

$$f_y = 18ye^{10x^2+9y^2}. \qquad \square$$

Example 2. Given $f(x, y) = \sqrt{1 - 8x^2 - 10y^2}$, find f_x and f_y.

Solution. It's probably easiest to rewrite $f(x, y) = (1 - 8x^2 - 10y^2)^{1/2}$. Now we use the power rule and the chain rule to obtain

$$f_x = \frac{1}{2}(1 - 8x^2 - 10y^2)^{-1/2} \cdot -16x = \frac{-8x}{\sqrt{1 - 8x^2 - 10y^2}}$$
$$f_y = \frac{1}{2}(1 - 8x^2 - 10y^2)^{-1/2} \cdot -10y = \frac{-10y}{\sqrt{1 - 8x^2 - 10y^2}} \qquad \square$$

Example 3. Given $f(x,y) = 2x\tan(10y)$, find f_x and f_y.

Solution. Here $f_x = 2\tan(10y)$. Remember that we are thinking of y as a constant, so $2\tan(10y)$ is all a constant being multiplied by x in this case. To find f_y, we think of $2x$ as being a constant multiplied by $\tan(10y)$. Then $f_y = 20x\sec^2(10y)$, accounting for the chain rule. □

Example 4. Given

$$f(x,y) = \frac{4x^2y^3}{y-10x},$$

find $f_x(1,-1)$.

Solution. Here we want to evaluate the partial derivative with respect to x at the point $(1,-1)$. We employ our favorite rule, the quotient rule. Note that you could use the product rule by rewriting the denominator with a negative exponent.

$$\begin{aligned}
f_x &= \frac{(y-10x)\cdot 8xy^3 + 10(4x^2y^3)}{(y-10x)^2} \\
f_x(1,-1) &= \frac{((-1)-10(1)\cdot 8(1)(-1)^3 + 10(4(1)^2((-1)^3)}{((-1)-10(1)^2} \\
&= \frac{48}{121}
\end{aligned}$$

It's easiest to wait to simplify after plugging in the numbers since that's something your calculator can do. □

Example 5. Given

$$z = \frac{xy^2}{x^2y^3+1},$$

find $\frac{\partial z}{\partial y}$.

Solution. This is just another application of the quotient rule.

$$\begin{aligned}
\frac{\partial z}{\partial y} &= \frac{(x^2y^3+1)\cdot 2xy - xy^2(3x^2y^2)}{(x^2y^3+1)^2} \\
&= \frac{2x^3y^4 + 2xy - 3x^3y^4}{(x^2y^3+1)^2} \\
&= \frac{2xy - x^3y^4}{(x^2y^3+1)^2}
\end{aligned}$$

Note that while simplifying your answer is not necessary for Loncapa, it does sometimes make it easier to input it with the correct syntax without cursing your computer. □