Overview

In this lesson we discuss higher order partial derivatives. If you know how to compute a partial derivative, you know how to compute any number of them.

Lesson

The key to this lesson is really more behind-the-scenes for this course. But just as we can talk about second derivatives in one-variable, we can talk about the second partial derivatives of a multivariate function. The main difference is that there are more possibilities since we can mix partials (first do x then y or vice versa). Let's start with the notation.

Notation. To denote the second partial derivative with respect to x, we write

$$\frac{\partial^2}{\partial x^2} \quad \text{or} \quad f_{xx}.$$

Similarly for the second partial derivative with respect to y. To denote the partial derivative with respect to y then with respect to x, we write

$$\frac{\partial^2}{\partial x \partial y} \quad \text{or} \quad f_{xy}$$

The behind-the-scenes thing is the following theorem.

Theorem. If given $z = f(x, y)$ has that f_{xx}, f_{yy}, f_{xy} and f_{yx} are all continuous, then

$$f_{xy} = f_{yx}$$

What this amounts to is that it doesn't matter what order we take our mixed partial derivatives in. As a result, we will just write f_{xy} for both taking the partial with respect to x first and with respect to y first.

Example 1. For the function $f(u, v) = 7u \ln(6uv) - 8u^2$, compute f_{uu}.

Solution. To do this we need to first compute f_{uu}. If we write $\ln(6uv) = \ln 6 + \ln u + \ln v$, it should be clear that $\frac{\partial}{\partial u} = \frac{1}{u}$.

$$\begin{aligned} f_u &= 7\ln(6uv) + 7u \cdot \frac{1}{u} - 16u \\ &= 7\ln(6uv) + 7 - 16u \end{aligned}$$

$$f_{uu} = \frac{7}{u} - 16$$ □

Example 2. Find the second partial derivatives for the given function.

$$f(x, y) = -\frac{3x \ln(3xy)}{4y}$$

Solution. We'll do this one using the product rule by writing $f(x,y) = -3x(4y)^{-1}\ln(3xy)$. It will also be helpful to note that

$$\ln(3xy) = \ln 3 + \ln x + \ln y. \tag{1}$$

Now in order to compute f_{xx}, f_{xy} and f_{yy}, we will first need to find f_x and f_y. Starting with f_x, we note that $(4y)^{-1}$ is a constant, then use the product rule on the remaining terms.

$$\begin{aligned}
f_x &= (4y)^{-1}\left[-3\ln(3xy) + (-3x)\cdot\frac{1}{x}\right] \\
&= (4y)^{-1}\left[-3\ln(3xy) - 3\right] \\
&= -\frac{3}{4}y^{-1}\ln(3xy) - \frac{3}{4}y^{-1}
\end{aligned}$$

To compute f_y we note that $-3x$ is a constant and use the product rule on the remaining terms.

$$\begin{aligned}
f_y &= -3x\left[-4(4y)^{-2}\ln(3xy) + (4y)^{-1}\cdot\frac{1}{y}\right] \\
&= -3x\left[-\frac{1}{4}y^{-2}\ln(3xy) + \frac{1}{4}y^{-2}\right] \\
&= \frac{3x}{4}y^{-2}\ln(3xy) - \frac{3}{4}xy^{-2}
\end{aligned}$$

Now we can finally compute the second partials.

$$\begin{aligned}
f_{xx} &= -\frac{3}{4}y^{-1}\cdot\frac{1}{x} \\
&= -\frac{3}{4xy}
\end{aligned}$$

$$\begin{aligned}
f_{yy} &= -\frac{6x}{4}y^{-3}\ln(3xy) + \frac{3x}{4}y^{-2}\cdot\frac{1}{y} + \frac{6}{4}xy^{-3} \\
&= -\frac{6x\ln(3xy)}{4y^3} + \frac{3x}{4y^3} + \frac{6x}{4y^3} \\
&= \frac{9x - 6x\ln(3xy)}{4y^3} \\
&= \frac{3x(2 - 3\ln(3xy))}{4y^3}
\end{aligned}$$

Now to compute f_{xy} we can either look at f_x or f_y and take the other derivative. We'll take the y derivative of f_x.

$$\begin{aligned}
f_{xy} &= \frac{3}{4}y^{-2}\ln(3xy) - \frac{3}{4}y^{-1}\cdot\frac{1}{y} + \frac{3}{4}y^{-2} \\
&= \frac{3\ln(3xy)}{4y^2} - \frac{3}{4y^2} + \frac{3}{4y^2} \\
&= \frac{3\ln(3xy)}{4y^2}
\end{aligned}$$

□