Overview

There is no new material for this lesson. Here, we explore a multitude of fun word problems that we can now do using the method of Lagrange multipliers from the previous lesson.

Examples

Example 1. You have exactly 24 hours to study for a final exam worth 1000 points, and without preparation you will get 200 points. You wasted all your time earlier to determine that your exam score will improve $x(38-x)$ points if you read lecture notes for x hours and $y(51-y)$ points if you solve review problems for y hours, but due to fatigue from your last minute cramming, you will lose $(x+y)^2$ points. What is the maximum score you can obtain?

Solution. First we need to parse through all the information here and find a function $f(x,y)$ which represents the score you will get on the exam, and a constraint function $g(x,y)$. Let's start with the constraint. You only have 24 hours to study which we're assuming you use up completely without stopping to sleep/eat/go to the bathroom, etc. So then our constraint will just be

$$g(x,y) = x + y = 24.$$

To figure out what f is isn't much harder. Reading for x hours will add $x(38-x)$ to your score, and solving review problems for y hours will add $y(51-y)$ to your score, and doing all this will take away $(x+y)^2$ from your score. Since we're assuming that you're already going to get 200 points, we have

$$f(x,y) = 200 + x(38-x) + y(51-y) - (x+y)^2.$$

Now that we have set up the equations, we proceed with what we learned in the previous lesson. We start by setting $f_x = \lambda g_x$ and $f_y = \lambda g_y$. So,

$$\begin{aligned} f_x &= 38 - 2x - 2(x+y) = \lambda \\ f_y &= 51 - 2y - 2(x+y) = \lambda. \end{aligned}$$

Notice that both $f_x = \lambda$ and $f_y = \lambda$. So setting these equal to each other, we get

$$\begin{aligned} 38 - 2x - \cancel{2(x+y)} &= 51 - 2y - \cancel{2(x+y)} \\ 38 - 2x &= 51 - 2y \\ 2y &= 13 + 2x \\ \boxed{y = \frac{13}{2} + x.} \end{aligned}$$

Plugging this into our constraint, we have

$$\begin{aligned} x + \frac{13}{2} + x &= 24 \\ 2x &= 24 - \frac{13}{2} \\ \boxed{x = \frac{35}{4}.} \end{aligned}$$

Using our equation for y above, we find that $y = \frac{13}{2} + \frac{35}{4} = \frac{61}{4}$.

Plugging these points into our function f, we get $f(35/4, 61/4) = 425.125$ as the maximum score possible. $\square$

Example 2. You place a termite on a circular heated plate whose temperature is modeled by

$$f(x, y) = y^2 - x^2 + 5$$

degrees Celsius, where x and y are in meters from the center of the plate. The termite walks along just the outer edge of the plate, which has a radius of 7 meters. (It's an obscenely large plate). What is the warmest the termite could get traveling along this path?

Solution. Since the termite is walking along the boundary of the plate, the distance from the termite to the center must always be 7, or the square of the distance must always be 49. That is, we have a constraint of

$$g(x, y) = x^2 + y^2 = 49$$

Next, setting $f_x = \lambda g_x$ and $f_y = \lambda g_y$,

$$-2x = \lambda 2x \tag{1}$$
$$2y = \lambda 2y \tag{2}$$

Canceling the 2 on each side in (2), we get $y = \lambda y$, which tells us that either $\lambda = 1$, or $y = 0$. Looking at (1), we have $-x = \lambda x$, which tells us either $x = 0$ or $\lambda = -1$. Notice that if $\lambda = -1$, then plugging this into (2) tells us $y = 0$ and plugging $\lambda = 1$ into (1) tells us $x = 0$. So the only solutions to these to equations are $x = 0$ or $y = 0$.

Let's imagine that we have superimposed the xy-plane onto this heated plate. Then a negative x or y value just corresponds to being on the left or bottom half of the plate, respectively. If $x = 0$, then from our constraint we have $y^2 = 49$, so $y = \pm 7$. And if $y = 0$ we have $x^2 = 49$, so $x = \pm 7$. Now computing f at these points, we get

$$f(0, \pm 7) = 49 - 0 + 5 = 54$$
$$f(\pm 7, 0) = 0 - 49 + 5 = -44.$$

Thus the maximum temperature the termite can find on the plate is 54 °C. Pay no mind to the fact that the termite would likely die. $\square$

Example 3. A fruit stand exclusively sells dragon fruit and guava. If the owner puts x pieces of dragon fruit and y guavas on the stand at the beginning of a day, it is estimated that he will make a profit of

$$P(x, y) = 5x^{3/2}y^{1/2}$$

dollars that day. If he can only put 130 total pieces of fruit on the stand per day, what is the maximum profit that the owner can make that day?

Solution. Since he can only put 130 pieces fruit on the stand, we have a constraint equation of $g(x, y) = x + y = 130$. Next,

$$P_x = \frac{15}{2}x^{1/2}y^{1/2} = \lambda = \lambda g_x$$
$$P_y = \frac{5}{2}x^{3/2}y^{-1/2} = \lambda = \lambda g_y.$$

Notice here $P_x = P_y$, so

$$\begin{aligned}
\frac{15}{2}x^{1/2}y^{1/2} &= \frac{5}{2}x^{3/2}y^{-1/2} \qquad (*)\\
\frac{15}{2}\cdot\frac{2}{5}x^{1/2}y &= x^{3/2}\\
3x^{1/2}y &= x^{3/2}\\
0 &= x^{3/2} - 3x^{1/2}\\
0 &= x^{1/2}\left(x - 3y\right),
\end{aligned}$$

which tells us that either $x = 0$ or $x = 3y$. If $x = 0$, then from our constraint equation it's easy to see that $y = 130$. If $x = 3y$, then

$$\begin{aligned}
3y + y &= 130\\
4y &= 130\\
y &= 32.5.
\end{aligned}$$

And since $x = 3y$, we get $x = 97.5$. Plugging these values into our profit function, we get a maximum profit of $5(97.5)^{3/2}(32.5)^{1/2}$, or about \$27442. This is a highly lucrative fruit stand. $\square$

Remark. In solving for x and y in $(*)$ here we started by multiplying both sides by $\frac{2}{5}y^{1/2}$. Another approach, and the one we did in class, is to just start by factoring. Recall that when factoring we pull out the *smallest* power of each common term, and to figure out what's left, we subtract the power we pulled out from the original power. That is, in this case

$$\begin{aligned}
0 &= \frac{15}{2}x^{1/2}y^{1/2} - \frac{5}{2}x^{3/2}y^{-1/2}\\
&= \frac{5}{2}x^{1/2}y^{-1/2}\big(3y^{1/2-(-1/2)} - x^{3/2-1/2}\big)\\
&= \frac{5}{2}x^{1/2}y^{-1/2}\big(3y - x\big).
\end{aligned}$$

Example 4. You're at it again making rectangular boxes in your spare time. This time it has a square base, the material for the bottom costs \$7/sq. ft., the top costs \$4/sq. ft., and the sides cost \$3/sq. ft. Find the box of greatest volume that you can make for \$167. Note that buying a cheaper box on Amazon is not an option.

Solution. Let x be the length of the sides of the base and y the height of the box. Then we have an equation for cost

$$\begin{aligned}
C(x,y) &= \underbrace{7x^2}_{\text{top}} + \underbrace{4x^2}_{\text{bottom}} + \underbrace{3\cdot 4xy}_{\text{4 sides}}\\
C(x,y) &= 11x^2 + 12xy = 167
\end{aligned}$$

This is our constraint, and we want to maximize $V = x^2y$. So we set $V_x = \lambda C_x$, $V_y = \lambda C_y$ and solve as usual.

$$V_x = 2xy = \lambda(22x + 12y) = \lambda C_y \tag{3}$$

$$V_y = x^2 = \lambda(12x) = \lambda C_x \tag{4}$$

Let's start with (3). We have

$$\begin{aligned} x^2 - 12\lambda x &= 0 \\ x(x - 12\lambda) &= 0, \end{aligned}$$

thus either $x = 0$ or $x = 12\lambda$. Well certainly $x \neq 0$ otherwise our box wouldn't hold much (having zero volume). So using $x = 12\lambda$ in (4),

$$\begin{aligned} 2xy &= \left(\frac{x}{12}\right)(22x + 12y) \\ 2xy &= \frac{11}{6}x^2 + xy \\ 0 &= xy - \frac{11}{6}x^2 \\ 0 &= x\left(y - \frac{11}{6}x\right). \end{aligned}$$

As we've mentioned, $x \neq 0$, so we have $y = \frac{11}{6}x$. Now using our cost equation,

$$\begin{aligned} 11x^2 + 12xy &= 167 \\ 11x^2 + 12x\left(\frac{11}{6}x\right) &= 167 \\ 11x^2 + 22x^2 &= 167 \\ 33x^2 &= 167 \\ x &= \sqrt{\frac{167}{33}}. \end{aligned}$$

So then $y = \frac{11}{6}\sqrt{167/33}$. And the maximum volume given the cost constraint is

$$V = \frac{11}{6}\left(\frac{167}{33}\right)^{3/2} \approx 20.87. \qquad \square$$