Overview

Matrices aren't just useful for efficiently solving systems of linear equations. We can actually do a lot of operations with matrices which have far-reaching applications. We discuss some of those operations and allude to a few applications in this lesson.

Lesson

Matrix operations

Just as we can add and multiply real numbers, there are operations that we can perform with matrices. We can add (or subtract) and multiply matrices under certain conditions. We can also multiply matrices by real numbers or raise them to integer powers. We discuss these below.

Before we get into the operations though, we should introduce some definitions and notation. We will usually use A and B or M and N to denote matrices. If A is a matrix which has m rows and n columns, we say that A is an $m \times n$ matrix, and the shorthand for this is $A_{m\times n}$. We will denote the (i,j)th entry – that is, the ith row and the jth column of A – by a_{ij}.

Matrix addition

In order to add (or subtract) two matrices, they must be of the same size. That is, if we want to perform $A \pm B$, we must have that A and B are both $m \times n$ matrices. Say

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}.$$

Then

$$A \pm B = \begin{bmatrix} a_{11} \pm b_{11} & a_{12} \pm b_{12} & a_{13} \pm b_{13} \\ a_{21} \pm b_{21} & a_{22} \pm b_{22} & a_{23} \pm b_{23} \\ a_{31} \pm b_{31} & a_{32} \pm b_{32} & a_{33} \pm b_{33} \end{bmatrix}$$

In other words, for addition, we just add A and B entry by entry. With repeated addition, we can talk about *scalar multiplication*. By scalar, we just mean a real number, i.e., not a matrix. It follows that to compute a scalar times a matrix, we do so entry by entry. So,

$$cA = \begin{bmatrix} c \cdot a_{11} & c \cdot a_{12} & c \cdot a_{13} \\ c \cdot a_{21} & c \cdot a_{22} & c \cdot a_{23} \\ c \cdot a_{31} & c \cdot a_{32} & c \cdot a_{33} \end{bmatrix}.$$

Example 1. Let $A = \begin{bmatrix} 1 & 2 & -2 \\ 4 & 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -7 & 10 & 3 \\ 6 & 1 & 4 \end{bmatrix}$. Then A and B are both 2×3 matrices, so we can compute

$$A + B = \begin{bmatrix} -6 & 12 & 1 \\ 10 & 4 & 5 \end{bmatrix}. \qquad \square$$

Matrix multiplication

The way matrix multiplication is defined is a little convoluted, but it's not bad once you get the hang of it. If A and B are matrices and we want to compute AB we need the number of columns of A to match the number of rows of B. That is, we need that A be $m \times n$ and B be $n \times k$. Note that we could have $m = n = k$, in which case we have square matrices, but this is certainly not necessary. With A and B as before, we show how to compute AB.

$$\begin{aligned} AB &= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \\ &= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ a_{21}b_{21} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} & a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} & a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33} \end{bmatrix} \end{aligned}$$

So how on earth can we remember to do this? It's helpful to color-code how to obtain each entry. Let's say that $C = AB$. Then in order to compute c_{11}, say, we use the first row of A and the first column of B.

$$\begin{bmatrix} \color{red}{a_{11}} & \color{red}{a_{12}} & \color{red}{a_{13}} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \color{red}{b_{11}} & b_{12} & b_{13} \\ \color{red}{b_{21}} & b_{22} & b_{23} \\ \color{red}{b_{31}} & b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} \color{red}{c_{11}} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

Or to compute c_{23}, we use the second row of A and the third column of B.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ \color{blue}{a_{21}} & \color{blue}{a_{22}} & \color{blue}{a_{23}} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \color{blue}{b_{13}} \\ b_{21} & b_{22} & \color{blue}{b_{23}} \\ b_{31} & b_{32} & \color{blue}{b_{33}} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & \color{blue}{c_{23}} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

To do this in practice, it's helpful to trace the appropriate row of A and column of B with your fingers and multiply the "matching" entries and add the result.

Example 2. Let $A = \left[\begin{smallmatrix} 3 & 2 \\ 1 & 1 \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} 9 & 2 \\ 6 & 1 \end{smallmatrix}\right]$. Compute AB and BA.

Solution. Following the multiplication rule described above, we have

$$\begin{aligned} AB &= \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 9 & 2 \\ 6 & 1 \end{bmatrix} = \begin{bmatrix} 3 \cdot 9 + 2 \cdot 6 & 3 \cdot 2 + 2 \cdot 1 \\ 1 \cdot 9 + 1 \cdot 6 & 1 \cdot 2 + 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} 39 & 8 \\ 15 & 3 \end{bmatrix} \\ BA &= \begin{bmatrix} 9 & 2 \\ 6 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 9 \cdot 3 + 2 \cdot 1 & 9 \cdot 2 + 2 \cdot 1 \\ 6 \cdot 3 + 1 \cdot 1 & 6 \cdot 2 + 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} 29 & 20 \\ 19 & 13 \end{bmatrix} \end{aligned}$$ □

Remark. Unlike multiplication of real numbers, matrix multiplication does not commute. That is, $AB \neq BA$ in general. In fact, sometimes BA is not even defined, even if AB is.

Example 3. Let

$$A = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 9 \\ 3 & 1 \\ 6 & 2. \end{bmatrix}$$

Compute AB and BA if they exist.

Solution. Writing the dimension of A and B, we have $A_{1\times 3}B_{3\times 2}$, so AB is defined, but $B_{3\times 2}A_{1\times 3}$ is undefined, since the inside dimensions don't match. Computing AB, we get

$$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 9 \\ 3 & 1 \\ 6 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 9 \end{bmatrix}$$

□

Remark. Notice that when we multiply $A_{1\times 3}B_{3\times 2}$ we get $AB_{1\times 2}$. This is always the case. So more generally, if we multiply $A_{m\times n}B_{n\times k}$, the result will be $C_{m\times k}$.

Just as repeated addition gets us scalar multiplication, we can use repeated multiplication to raise matrices to integer powers. That is

$$A^n = \underbrace{A \cdot A \cdots A}_{n \text{ times}}$$

Example 4. Given

$$A = \begin{bmatrix} 3 & -1 & 4 \\ 2 & 1 & 1 \\ 6 & 3 & 7 \end{bmatrix},$$

compute A^2.

Solution.

$$A^2 = \begin{bmatrix} 3 & -1 & 4 \\ 2 & 1 & 1 \\ 6 & 3 & 7 \end{bmatrix} \begin{bmatrix} 3 & -1 & 4 \\ 2 & 1 & 1 \\ 6 & 3 & 7 \end{bmatrix} = \begin{bmatrix} 31 & 8 & 39 \\ 14 & 2 & 16 \\ 66 & 18 & 76 \end{bmatrix}$$

□

Applications

Why do we care about any of this? In the context of MA 16020, probably for the sole reason that it appears on the homework and exams. Nonetheless, you don't have to look long before finding matrix multiplication in the real world. It appears in

- Markov chains, which are studied in probability. A famous example being the PageRank algorithm used by Google.
- Linear programming. Many problems in operations research can be realized as an optimization problem of linear equations. Matrix multiplication is used when finding a solution.
- Graphics. If you represent a vector in a graph (just think of an arrow pointing to an ordered pair in the xy-plane), you can rotate and scale the vector using matrix multiplication. For example, if we want to rotate the vector $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ by an angle θ, we can do this by multiplying

  $$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix}.$$

  We can scale the same vector by multiplying by

  $$\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax \\ ay \end{bmatrix}.$$