Overview

In this lesson we restrict ourselves to square matrices in order to discuss inverses of matrices in both the 2×2 and 3×3 cases. The method we use in the 3×3 case can be applied to any $n \times n$ matrix, albeit much more monotonous.

Lesson

Matrix inverses

We start by defining the identity matrix and the inverse of a matrix.

Definition. The $n \times n$ *identity matrix*, denoted I_n or just I when the size is understood, is the matrix who has 1s along the main diagonal and 0s elsewhere.

Definition. If A and B are square matrices such that $AB = BA = I$, then B is the *inverse* of A, denoted A^{-1}.

It turns out that a matrix always commutes with its inverse (that is, $AA^{-1} = A^{-1}A$), so if we find *square* matrices with $AB = I$, then we know immediately that $BA = I$.

Remark. It is possible for non-square matrices to have $AB = I$ but $BA \neq I$. In fact, if A and B are not square matrices, then AB and BA have different sizes, or even worse, BA may not be defined.

2×2 Inverses

There is a convenient formula for finding the inverse of a 2×2 matrix, so we start with that. Given an invertible matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

$$\boxed{A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}} \tag{1}$$

Example 1. Find the inverse of

$$A = \begin{bmatrix} -\frac{1}{11} & \frac{2}{11} \\ -\frac{5}{22} & -\frac{1}{22} \end{bmatrix}.$$

Solution. This may look really gross at first, but we'll see it makes the answer nice and clean. This will be typical of problems you come across in Loncapa. Now using the formula in (1), we get

$$A^{-1} = \frac{1}{\left(-\frac{1}{11}\right)\left(-\frac{1}{22}\right) - \left(\frac{1}{11}\right)\left(-\frac{5}{22}\right)}\begin{bmatrix} -\frac{1}{22} & -\frac{2}{11} \\ \frac{5}{22} & -\frac{1}{11} \end{bmatrix} = 22\begin{bmatrix} -\frac{1}{22} & -\frac{2}{11} \\ \frac{5}{22} & -\frac{1}{11} \end{bmatrix} = \begin{bmatrix} -1 & -4 \\ 5 & -2 \end{bmatrix} \qquad \square$$

3×3 Inverses

The process for finding the inverse of a 3×3 matrix B is a bit more complicated. There's no convenient formula.

How to find a $n \times n$ inverse

Given an invertible matrix $A_{n\times n}$, to find A^{-1}, we do the following.

1. Set up an augmented matrix $[A \mid I_n]$.
2. Perform elementary row operations until you obtain an augmented matrix $[I_n \mid B]$.
3. The matrix $B = A^{-1}$.

Example 2. Given the invertible matrix

$$B = \begin{bmatrix} -\frac{3}{25} & -\frac{4}{75} & -\frac{1}{150} \\ \frac{4}{25} & -\frac{1}{25} & \frac{3}{25} \\ \frac{1}{15} & \frac{1}{15} & \frac{2}{15} \end{bmatrix}$$

find B^{-1}.

Solution. We start by augmenting B with the 3×3 identity matrix. If you don't want to work with fractions, then it is best to start by clearing denominators for your first elementary row operations.

$$\left[\begin{array}{ccc|ccc} -\frac{3}{25} & -\frac{4}{75} & -\frac{1}{150} & 1 & 0 & 0 \\ \frac{4}{25} & -\frac{1}{25} & \frac{3}{25} & 0 & 1 & 0 \\ \frac{1}{15} & \frac{1}{15} & \frac{2}{15} & 0 & 0 & 1 \end{array}\right] \xrightarrow{150R_1, 25R_2, 15R_3} \left[\begin{array}{ccc|ccc} -18 & -8 & -1 & 150 & 0 & 0 \\ 4 & -1 & 3 & 0 & 25 & 0 \\ 1 & 1 & 2 & 0 & 0 & 15 \end{array}\right]$$

$$\xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 0 & 15 \\ 4 & -1 & 3 & 0 & 25 & 0 \\ -18 & -8 & -1 & 150 & 0 & 0 \end{array}\right] \xrightarrow[18R_1+R_3]{-4R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 0 & 15 \\ 0 & -5 & -5 & 0 & 25 & -60 \\ 0 & 10 & 35 & 150 & 0 & 270 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{5}R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 0 & 15 \\ 0 & 1 & 1 & 0 & -5 & 12 \\ 0 & 10 & 35 & 150 & 0 & 270 \end{array}\right] \xrightarrow[-10R_2+R_3]{-R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 5 & 3 \\ 0 & 1 & 1 & 0 & -5 & 12 \\ 0 & 0 & 25 & 150 & 50 & 150 \end{array}\right]$$

$$\xrightarrow{\frac{1}{25}R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 5 & 3 \\ 0 & 1 & 1 & 0 & -5 & 12 \\ 0 & 0 & 1 & 6 & 2 & 6 \end{array}\right] \xrightarrow[-R_3+R_2]{-R_3+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -6 & 3 & -3 \\ 0 & 1 & 0 & -6 & -7 & 6 \\ 0 & 0 & 1 & 6 & 2 & 6 \end{array}\right]$$

So,

$$B^{-1} = \begin{bmatrix} -6 & 3 & -3 \\ -6 & -7 & 6 \\ 6 & 2 & 6 \end{bmatrix}.$$

□

Solving systems of equations

We've already discussed how to solve a system of equations using row reduction of matrices, but we can interpret a system of equations in a slightly different way. Say we want to solve the system

$$\begin{cases} ax+by+cz &= j \\ dx+ey+fz &= k \\ gx+hy+iz &= l \end{cases}$$

We can translate this to the matrix equation $AX = B$:

$$\underbrace{\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} j \\ k \\ l \end{bmatrix}}_{B}$$

In particular, if A is an invertible matrix, we can solve for X, by multiplying both sides by A^{-1}. That is, $X = A^{-1}B$. This is the matrix version of "dividing" both sides by A. It is important to note that we cannot divide matrices.

Remark. Recall that matrix multiplication does *not* commute. So it is important that we keep track of which side we multiply matrices on. To get X by itself, we multiplied on the *left* by A^{-1}, so we obtain $A^{-1}B$ on the other side of the equation.

Why does this work? Writing the details of what we have done:

$$\begin{aligned} AX &= B \\ A^{-1}AX &= A^{-1}B \\ IX &= A^{-1}B \\ X &= A^{-1}B \end{aligned}$$

We start by multiplying both sides by A^{-1}, which preserves equality. At the end, we are using the fact that $IX = X$. This is why we call I the identity matrix. It plays the same role as the number 1 does for real numbers. Let's see this in action with a simple example.

Example 3. Solve the system of equations

$$\begin{cases} 8x+y &= -36 \\ 5x+8y &= 7 \end{cases}$$

Solution. We start by translating this into the matrix equation

$$\begin{bmatrix} 8 & 1 \\ 5 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -36 \\ 7 \end{bmatrix}$$

Now computing

$$\begin{bmatrix} 8 & 1 \\ 5 & 8 \end{bmatrix}^{-1} = \frac{1}{64-5} \begin{bmatrix} 8 & -1 \\ -5 & 8 \end{bmatrix} = \frac{1}{59} \begin{bmatrix} 8 & -1 \\ -5 & 8 \end{bmatrix}.$$

Finally,

$$X = \frac{1}{59} \begin{bmatrix} 8 & -1 \\ -5 & 8 \end{bmatrix} \begin{bmatrix} -36 \\ 7 \end{bmatrix} = \frac{1}{59} \begin{bmatrix} -295 \\ 236 \end{bmatrix} = \begin{bmatrix} -5 \\ 4 \end{bmatrix}.$$ $\square$

Remark. In practice, by hand this is more work than the methods described in previous lessons. Moreover, matrices are not guaranteed to have an inverse, so this will not always work. It is good in theory, and some homework problems are set up to make it the most advantageous method.