**Instructions.** Show all work, with clear logical steps. No work or hard-to-follow work will lose points.

**Problem 1.** (6 points) Find and classify all critical points of the function

$$f(x, y) = 54x^4 + 64x + \frac{16}{3}y^3 - y + 2.$$

*Solution.* We start by taking the first partial derivatives:

$$\begin{aligned} f_x &= 216x^3 + 64 \\ f_y &= 16y^2 - 1. \end{aligned}$$

Setting $f_x = 0$ gives

$$\begin{aligned} 216x^3 + 64 &= 0 \\ 216x^3 &= -64 \\ x^3 &= \frac{-64}{216} \\ x &= -\frac{4}{6} \\ &= -\frac{2}{3} \end{aligned}$$

And setting $f_y = 0$ gives

$$\begin{aligned} 16y^2 - 1 &= 0 \\ 16y^2 &= 1 \\ y^2 &= \frac{1}{16} \\ y &= \pm\frac{1}{4}. \end{aligned}$$

Thus we have critical points at $(-2/3, 1/4)$ and $(-2/3, -1/4)$. Calculating the second partial derivatives,

$$\begin{aligned} f_{xx} &= 648x^2 \\ f_{yy} &= 32y \\ f_{xy} &= 0 \end{aligned}$$

Then the discriminant is

$$D = (648x^2)(32y) - 0.$$

At the point $(-2/3, 1/4)$ we have $D = (648(-2/3)^2)(32(1/4)) > 0$ and $f_{xx} = 648(-2/3)^2 > 0$, which gives us a local min. And at the point $(-2/3, -1/4)$, we have $D = (648(-2/3)^2)(32(-1/4)) < 0$, which tells us that this is a saddle point. $\square$

**Problem 2.** (4 points) Given the information in the table below, find and classify any critical points for the function $g(x, y)$.

| $(a, b)$ | $g(a, b)$ | $g_x(a, b)$ | $g_y(a, b)$ | $g_{xx}(a, b)$ | $g_{xy}(a, b)$ | $g_{yy}(a, b)$ |
|---|---|---|---|---|---|---|
| $(0, 1)$ | 0 | 3 | 0 | 0 | $-2$ | 4 |
| $(4, 3)$ | $-3$ | 0 | 0 | $-1$ | 2 | $-6$ |
| $(2, 7)$ | 15 | 0 | 0 | 4 | 5 | 8 |
| $(5, 6)$ | 4 | 0 | 0 | 3 | 5 | 2 |

*Solution.* This is an easy application of finding critical points and using the second derivative test. The point $(0, 1)$ isn't even a critical point since $g_x(0, 1) = 3 \neq 0$. The other three points are critical points since we have $g_x = g_y = 0$.

For the point $(4, 3)$, we compute

$$D = (-1)(-6) - (2)^2 = 6 - 4 > 0.$$

Since $D > 0$, we look at $g_{xx}(4, 3) = -1 < 0$, so we have a local maximum at $(4, 3)$.

For the point $(2, 7)$, we have

$$D = (4)(8) - (5)^2 = 32 - 25 > 0,$$

and $g_{xx} = 4 > 0$, so we have a local minimum at $(2, 7)$.

For the point $(5, 6)$, we find

$$D = (3)(2) - (5)^2 = 6 - 25 < 0,$$

which means that we have a saddle point at $(5, 6)$. □

**Second Derivative Test.** Suppose $(a, b)$ is a critical point of $f$ and the second partial derivatives exist and are continuous. Let

$$D = D(a, b) = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2 .$$

Then

(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.

(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.

(c) If $D < 0$, then $f(a, b)$ is a saddle point.

(d) If $D = 0$ then the test is inconclusive.