Instructions. Show all work, with clear logical steps. No work or hard-to-follow work will lose points.

Problem 1. (4 points) Find the maximum value of the function

$$f(x,y) = e^{8xy}$$

subject to the constraint $x^2 + y^2 = 100$. Assume x and y are both positive.

Solution. This was Example 1 from Lesson 25. Here $g(x,y) = x^2 + y^2 = 100$. Using Lagrange multipliers, we have

$$f_x = 8ye^{8xy} = \lambda 2x = \lambda g_x \quad (1)$$

$$f_y = 8xe^{8xy} = \lambda 2y = \lambda g_y \quad (2)$$

$$g(x,y) = x^2 + y^2 = 100 \quad (3)$$

Solving (1) for λ, we get

$$\lambda 2x = 8ye^{8xy}$$
$$\lambda = \frac{4}{x} ye^{8xy}.$$

Note that we are allowed to divide by x because $x > 0$ by assumption (in particular, $x \neq 0$). Plugging this into λ in (2),

$$8xe^{8xy} = \lambda 2y$$
$$8xe^{8xy} = \frac{4}{x} ye^{8xy} 2y$$
$$8x = \frac{8}{x} y^2$$
$$x^2 = y^2.$$

Now we use (3):

$$x^2 + y^2 = 100$$
$$x^2 + x^2 = 100$$
$$2x^2 = 100$$
$$x^2 = 50.$$

Since x and y are both positive (by assumption), $x^2 = y^2$ implies that $x = y$. Since we found that $x^2 = 50$, this means that $xy = 50$. Thus the maximum value of f is $e^{8 \cdot 50} = e^{400}$. □

Problem 2. (6 points) A rectangular box with a square base is to be constructed from material that costs \$5/ft^2 for the bottom, \$3/ft^2 for the top and \$5/ft^2 for he sides. Find the box of greatest volume that can be constructed for \$167. Round your answer to 2 decimals.

Solution. Let x be the length of the sides of the base and y the height of the box. Then we have an equation for cost

$$\begin{aligned} C(x,y) &= \underbrace{3x^2}_{\text{top}} + \underbrace{5x^2}_{\text{bottom}} + \underbrace{4 \cdot 5xy}_{\text{4 sides}} \\ C(x,y) &= 8x^2 + 20xy = 167 \end{aligned}$$

This is our constraint, and we want to maximize $V = x^2y$. So we set $V_x = \lambda C_x$, $V_y = \lambda C_y$ and solve as usual.

$$V_x = 2xy = \lambda(16x + 20y) = \lambda C_y \tag{4}$$

$$V_y = x^2 = \lambda(20x) = \lambda C_x \tag{5}$$

Let's start with (4). We have

$$\begin{aligned} x^2 - 20\lambda x &= 0 \\ x(x - 20\lambda) &= 0, \end{aligned}$$

thus either $x = 0$ or $x = 20\lambda$. Well certainly $x \neq 0$ otherwise our box wouldn't hold much (having zero volume). So using $x = 20\lambda$ in (5),

$$\begin{aligned} 2xy &= \left(\frac{x}{20}\right)(16x + 20y) \\ 40y &= 16x + 20y \\ 20y &= 16x \\ y &= \frac{4}{5}x \end{aligned}$$

As we've mentioned, $x \neq 0$, so we have $y = \frac{4}{5}x$. Now using our cost equation,

$$\begin{aligned} 8x^2 + 20xy &= 167 \\ 8x^2 + 20x\left(\frac{4}{5}x\right) &= 167 \\ 8x^2 + 16x^2 &= 167 \\ 24x^2 &= 167 \\ x &= \sqrt{\frac{167}{24}}. \end{aligned}$$

So then $y = \frac{4}{5}\sqrt{167/24}$. And the maximum volume given the cost constraint is

$$V = \frac{4}{5}\left(\frac{167}{24}\right)^{3/2} \approx 14.68.$$

□