Instructions. Show all work, with clear logical steps. No work or hard-to-follow work will lose points.

Problem 1. (3 points) Determine whether the following differential equations is linear.

(a.) $y' + (x^2 + \cos x)y = x$

(b.) $yy' = x$

(c.) $y' - xy = x^2y + 3$

Solution.

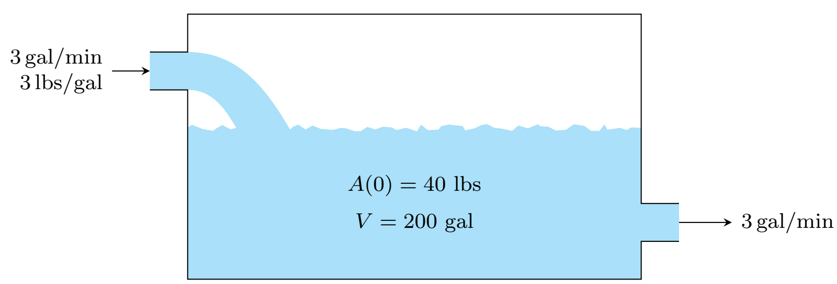
(a) Yes

(b) No

(c) Yes $\square$

Problem 2. (4 points) A 400-gallon tank initially contains 200 gallons of water and 40 pounds of salt. A solution containing 3 pounds of salt per gallon enters the tank at a rate of 3 gallons per minute, and the well-stirred mixture flows out of the tank at a rate of 3 gallons per minute. How many pounds of salt are in the tank 10 minutes later? **Round your answer to 2 decimal places.**

Solution. The picture below represents the situation we're after.



Notice that the volume is not changing. Now let $A(t)$ represent the number of pounds of salt in the tank at time t. Then

$$\begin{aligned}\frac{dA}{dt} &= 3\frac{\text{gal}}{\text{min}} \cdot 3\frac{\text{lbs}}{\text{gal}} - 3\frac{\text{gal}}{\text{min}} \cdot \frac{A(t)}{200}\frac{\text{lbs}}{\text{gal}} \\ &= 9 - \frac{3}{200}A.\end{aligned}$$

So

$$\frac{dA}{dt} + \frac{3}{200}A = 9.$$

This is first order linear with $p(t) = 3/200$, so that $\int p(t)\,dt = \frac{3t}{200}$, and our integrating factor is

$$\mu(t) = e^{3t/200}.$$

So a solution is given by

$$\begin{aligned} Ae^{3t/200} &= \int 9e^{3t/200}\,dt \\ Ae^{3t/200} &= \frac{200}{3}\cdot 9e^{3t/200} + C \\ Ae^{3t/200} &= 600e^{3t/200} + C. \end{aligned} \tag{$*$}$$

We solve for C using $A(0) = 40$. So $40 = 600 + C$, which means that $C = -560$. Now multiplying both sides of $(*)$ by $e^{-3t/200}$, we get

$$A(t) = 600 - 560e^{-3t/200}.$$

Finally, computing $A(t)$, we get

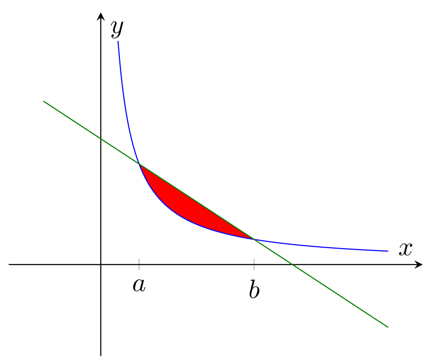
$$\begin{aligned} A(10) &= 600 - 560e^{-30/200} \\ &\approx 118.00. \end{aligned}$$

□

Problem 3. (3 points) Set up but **do not compute** an integral that gives the area of the region R bounded by

$$f(x) = \frac{16}{x} \quad \text{and} \quad g(x) = -9x + 30.$$

Solution. It helps to draw a picture.



We need to determine what the values of a and b are. To do this, we set $f(x) = g(x)$.

$$\begin{aligned} \frac{16}{x} &= -9x + 30 \\ 16 &= -9x^2 + 30x \\ 0 &= 9x^2 - 30x + 16. \end{aligned}$$

Using the quadratic formula, we get

$$\begin{aligned} x &= \frac{30 \pm \sqrt{(-30)^2 - 4(9)(16)}}{2(9)} \\ &= \frac{30 \pm 18}{18} \\ &= \frac{2}{3}, \frac{8}{3} \end{aligned}$$

Finally the area of the red region is given by

$$\int_{2/3}^{8/3} \left[(-9x + 30) - \frac{16}{x} \right] dx.$$

$\square$