Instructions. Show all work, with clear logical steps. No work or hard-to-follow work will lose points.

1.) (6 points) Consider the function

$$F(x,y) = \sqrt{1 - x^2 - y^2}.$$

(a) Sketch $F(x,y)$.

(b) State the domain of F.

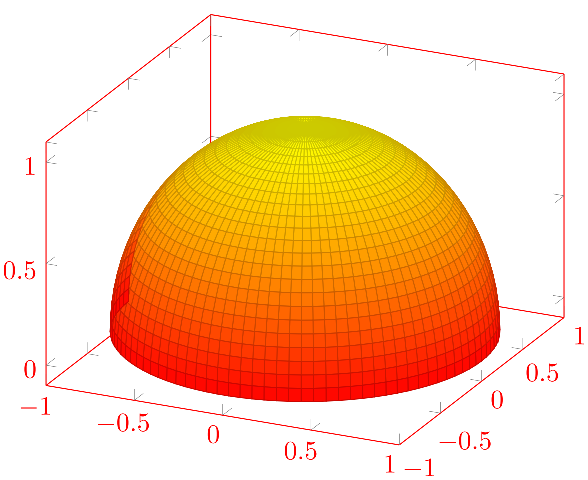
(c) State the range of F.

Solution. (a) With $z = F(x,y)$, we have

$$\begin{aligned} z &= \sqrt{1-x^2-y^2} \\ z^2 &= 1 - x^2 - y^2 && (z > 0) \\ 1 &= x^2 + y^2 + z^2 && (z > 0) \end{aligned}$$

So this is the upper half sphere.

Graph of $F(x,y)$



(b) We can't take the square root of a negative, so we need $1 - x^2 - y^2 \geq 0$, or $x^2 + y^2 \leq 1$. So the domain of F is $\{\,(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\,\}$.

(c) The smallest a square root can be is zero, which is attained when $x^2 + y^2 = 1$. Then F reaches its maximum when $x^2 + y^2$ is minimized, which is when $x = y = 0$. So the range of F is $[0,1]$. □

2.) (10 points) Compute the following limits, or prove the limit does not exist.

(a) $$\lim_{(x,y)\to(0,0)} \frac{4xy}{3x^2+y^2}$$

(b) $$\lim_{(x,y)\to(-1,1)} \frac{2x^2 - xy - 3y^2}{x+y}$$

Solution. (a) Along the path $y = mx$, we have

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} \frac{4xy}{3x^2+y^2} &= \lim_{(x,mx)\to(0,0)} \frac{4x(mx)}{3x^2+(mx)^2} \\
&= \lim_{(x,mx)\to(0,0)} \frac{4mx^2}{3x^2+m^2x^2} \\
&= \lim_{(x,mx)\to(0,0)} \frac{4m\cancel{x^2}}{\cancel{x^2}(3+m^2)} \\
&= \lim_{(x,mx)\to(0,0)} \frac{4m}{3+m^2} \\
&= \frac{4m}{3+m^2},
\end{aligned}$$

which depends on m, so the limit does not exist.

(b) The limit here does exist.

$$\begin{aligned}
\lim_{(x,y)\to(-1,1)} \frac{2x^2-xy-3y^2}{x+y} &= \lim_{(x,y)\to(-1,1)} \frac{\cancel{(x+y)}(2x-3y)}{\cancel{x+y}} \\
&= \lim_{(x,y)\to(-1,1)} (2x-3y) \\
&= 2(-1)-3(1) \\
&= 5.
\end{aligned}$$

□

3.) (4 points) Compute $\partial f/\partial y$ for

$$f(x, y) = 3y^2 \sinh(\sqrt{x^3-1}).$$

Solution. Notice that $\sinh(\sqrt{x^3-1})$ is a function of x only, so

$$\partial f/\partial y = 6y \sinh(\sqrt{x^3-1}).$$

□