1. (7 points) Given two series $\sum a_n$ and $\sum b_n$, which of the following statements is true?

I. If the sequence $\{a_n\}$ converges to $\frac{1}{2}$, then the series $\sum a_n$ converges.

II. If $\sum a_n$ and $\sum b_n$ both converge, then $\sum (a_n b_n) = \left(\sum a_n\right)\left(\sum b_n\right)$

III. If the series $\sum a_n$ converges, then the sequence $\{a_n\}$ also converges.

A. I. only

B. II. only

C. III. only

D. I. and III.

E. II. and III.

2. (7 points) For what values of p does the integral

$$\int_e^\infty \frac{1}{x(\ln x)^p}\,dx$$

converges?

$u = \ln x$

$du = \frac{1}{x}\,dx$

$\int_1^\infty \frac{du}{u^p}$

A. For $p \neq 0$

B. For $p < 0$

C. For $p > 1$

D. For $|p| < 1$

E. For all p

3. (7 points) The masses m_i are located at the points P_i. Find the moment M_x.

$$m_1 = 2,\ m_2 = 5,\ m_3 = 3; \quad P_1(2,-3),\ P_2(-3,1),\ P_3(3,5)$$

A. -10

B. -2

C. 0

D. 14

E. 30

$2 \cdot -3 + 5 \cdot 1 + 3 \cdot 5$

$= 14$

4. (7 points) Find the limit of the following sequence if it converges.

$$a_n = n\sin(\pi/n)$$

A. -1

B. 0

C. 1

D. π

E. The sequence does not converge.

$x \sin\left(\frac{\pi}{x}\right) = \dfrac{\sin \frac{\pi}{x}}{1/x}$

$\xrightarrow{LH} \dfrac{-\frac{\pi}{x^2}\cos(\pi/x)}{-1/x^2}$

$= \pi \cos(\pi/x) \longrightarrow \pi$

5. (14 points) The curve $y = \frac{1}{4}x^2 - \frac{1}{2}\ln x$ is rotated about the y-axis for $1 \le x \le 2$. Find the area of the resulting surface.

$$S = \int_1^2 2\pi x \, ds, \qquad ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$\left(\frac{dy}{dx}\right)^2 = \left(\frac{1}{2}x - \frac{1}{2x}\right)^2 = \frac{1}{4}x^2 - \frac{1}{2} + \frac{1}{4x^2}$$

$$\Rightarrow ds = \sqrt{1 + \frac{1}{4}x^2 - \frac{1}{2} + \frac{1}{4x^2}} = \sqrt{\frac{1}{4}x^2 + \frac{1}{2} + \frac{1}{4x^2}} = \sqrt{\left(\frac{1}{2}x + \frac{1}{2x}\right)^2} = \frac{1}{2}x + \frac{1}{2x}$$

$$S = \int_1^2 2\pi x\left(\frac{1}{2}x + \frac{1}{2x}\right) dx$$

$$= 2\pi \int_1^2 \left(\frac{x^2}{2} + \frac{1}{2}\right) dx$$

$$= 2\pi \left(\frac{x^3}{6} + \frac{1}{2}x \Big|_1^2\right)$$

$$= 2\pi \left[\left(\frac{8}{6} + \frac{2}{2}\right) - \left(\frac{1}{6} + \frac{1}{2}\right)\right]$$

$$= 2\pi \left[\frac{14}{6} - \frac{4}{6}\right]$$

$$= 2\pi \cdot \frac{10}{6}$$

$$= \boxed{\frac{10\pi}{3}}$$

6. (8 points) Set up, but **do not evaluate**, an integral to find the exact length of the curve $y = \ln(\cos x)$ for $0 \le x \le \pi/3$. Simplify your integrand as much as possible.

$$L = \int_0^{\pi/3} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$\frac{dy}{dx} = \frac{-\sin x}{\cos x} = -\tan x \Rightarrow \left(\frac{dy}{dx}\right)^2 = \tan^2 x$$

$$\Rightarrow L = \int_0^{\pi/3} \sqrt{1 + \tan^2 x}\, dx$$

$$= \int_0^{\pi/3} \sec x\, dx$$

7. (6 points) Let $a_n = f(n)$. Then the conclusion to the integral test is that the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if $\int_1^{\infty} f(x)\,dx$ is convergent. What are the three assumptions for the function f?

1. f is continuous
2. f is positive on $[1, \infty)$
3. f is decreasing

8. (16 points) Determine whether the series

$$\sum_{k=2}^{\infty} \frac{\ln k}{k^2}$$

converges. Fully justify your answer, including why any tests for convergence/divergence you use are valid.

Consider $f(x) = \frac{\ln x}{x^2}$, on $[2, \infty)$. Clearly f is positive and continuous on the interval $[2,\infty)$. To see decreasing

$$f'(x) = \frac{x^2 \cdot \frac{1}{x} - 2x\ln x}{x^4} = \frac{x - 2x\ln x}{x^4} = \frac{x(1-2\ln x)}{x^4},$$

which is equal to 0 when $x=0$, and $\ln x = \frac{1}{2} \Rightarrow x = e^{1/2} < 2 < e$

and $f'(e) = \frac{\cdot(1-2\ln e)}{4} < 0$ since $\ln e = 1$ and $1-2<0$.

So integral test applies.

$$\int_2^{\infty} \frac{\ln x}{x^2}\, dx \qquad \begin{matrix} u = \ln x & dv = \frac{1}{x^2}\, dx \\ du = \frac{1}{x} dx & v = \frac{-1}{x} \end{matrix}$$

$$\lim_{t\to\infty} \left[-\frac{1}{x}\ln x + \int \frac{1}{x^2} dx \right]_2^t = \lim_{t\to\infty} -\frac{1}{x}\ln x - \frac{1}{x}\Big|_2^t$$

$$= \lim_{t\to\infty} \left(\frac{-1}{t}\ln t - \frac{1}{t}\right) - \left(-\frac{1}{2}\ln 2 - \frac{1}{2}\right)$$

$$= \lim_{t\to\infty} \Big(\underbrace{\frac{-1}{t}}_{\downarrow 0} - \underbrace{\frac{1}{t}}_{\downarrow 0} \Big) - \left(-\frac{1}{2}\ln 2 - \frac{1}{2}\right) < \infty \Rightarrow$$ Series converges by Integral Test.

9. (14 points) Determine whether the series

$$\sum_{k=0}^{\infty} \frac{\sqrt[5]{k}}{\sqrt[5]{k^7+3k+4}}$$

converges. Fully justify your answer, including why any tests for convergence/divergence you use are valid.

$$b_k = \frac{k^{1/5}}{k^{7/5}} = \frac{1}{k^{6/5}}$$

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \frac{k^{1/5}}{(k^7+3k+4)^{1/5}} \cdot \frac{k^{6/5}}{1} = \lim_{k\to\infty} \frac{k^{7/5}}{(k^7+3k+4)^{1/5}}$$

$$= \lim_{k\to\infty} \frac{k^{7/5}}{k^{7/5}\left(1+\frac{3}{k^6}+\frac{4}{k^7}\right)^{1/5}} \Rightarrow \frac{1}{(1+0+0)^{1/5}} = 1$$

$\Rightarrow$ Converges by LCT.

10. (14 points) Consider the series

$$\sum_{n=0}^{\infty} \frac{(x-3)^n}{7^{2n}}.$$

(a) (7 points) Find the values x for which the series converges.

Need $\left|\frac{x-3}{49}\right| < 1$

$|x-3| < 49$

$-49 < x-3 < 49$

$\boxed{-46 < x < 52}$

(b) (7 points) Find the sum of the series for those values of x.

$$\sum_{n=0}^{\infty} \left(\frac{x-3}{49}\right)^n = \frac{1}{1-\frac{x-3}{49}}$$

$$= \frac{1}{\frac{49-(x-3)}{49}}$$

$$= \boxed{\frac{49}{52-x}}$$