1. (7 points) Suppose $\sum a_n$ and $\sum b_n$ are series, with $b_n > 0$ for all n. Which of the following must be true?

I. If $\sum a_n$ converges, then $\sum |a_n|$ also converges.

II. If $\sum b_n$ converges, then $\sum (-1)^{n-1} b_n$ also converges.

III. If $\lim_{n\to\infty} b_n = 0$, then $\sum (-1)^{n-1} b_n$ converges.

A. I only.

B. II only.

C. III only.

D. II and III only.

E. I, II and III.

2. (7 points) Suppose a series $\sum a_n$ is defined recursively as follows

$$a_1 = 1, \quad a_{n+1} = \frac{2 + \cos n}{\sqrt{n}} a_n.$$

Which of the following statements is true?

A. The series converges absolutely.

B. The series converges conditionally.

C. The series diverges.

D. There is not enough information to test for convergence.

E. This answer is incorrect.

3. (7 points) Evaluate the sum

$$1 - \ln 2 + \frac{(\ln 2)^2}{2!} - \frac{(\ln 2)^3}{3!} + \cdots$$

A. $-\ln 2$

B. $\ln 2$

C. $\frac{1}{2}$

D. 2

E. The series does not converge.

4. (7 points) Suppose a parametric curve is given by functions $x(t)$, $y(t)$ with

$$\frac{dx}{dt} = 1 - \frac{1}{t} \quad \text{and} \quad \frac{dy}{dt} = 1 + \frac{1}{t}.$$

Find $\frac{d^2y}{dy^2}$.

A. -1

B. $\dfrac{-1/t^2}{1-1/t}$

C. $\dfrac{2}{t^2(t-1)}$

D. $\dfrac{-2t}{(t-1)^3}$

E. More information is needed to be able to compute $\frac{d^2y}{dx^2}$.

5. (12 points) Determine whether the following series converge. Fully justify your response.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{n \cos n\pi}{2^n}$

$$\left| \frac{(n+1)\cos[(n+1)\pi]}{2^{n+1}} \cdot \frac{2^n}{n\cos(n\pi)} \right| = \left|\frac{n+1}{2n}\right| \left| \frac{\cos[(n+1)\pi]}{\cos(n\pi)} \right|$$

$$= \frac{1 + 1/n}{2} \xrightarrow[n\to\infty]{} \frac{1}{2} < 1 \Rightarrow \text{converges by Ratio Test}$$

(b) $\displaystyle\sum_{n=0}^{\infty} ne^{-n}$

$$\left| \frac{(n+1)\, e^{-(n+1)}}{n\ e^{-n}} \right| = \frac{n+1}{n} \cdot \frac{e^n}{e^n \cdot e}$$

$$= \frac{1 + 1/n}{1} \cdot \frac{1}{e} \xrightarrow[n\to\infty]{} \frac{1}{e} < 1$$

$\Rightarrow$ converges by Ratio test

6. (12 points) Determine whether the following series are conditionally convergent, absolutely convergent or divergent. Fully justify your response.

(a) $\displaystyle\sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{(\ln k)^k}$

$$\sqrt[k]{\left|\frac{(-1)^{k-1}}{(\ln k)^k}\right|} = \sqrt[k]{\frac{1}{(\ln k)^k}} = \frac{1}{\ln k} \xrightarrow[k\to\infty]{} 0 < 1$$

$\Rightarrow$ converges by Root Test

(b) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$

$$\sqrt[n]{\left(\frac{n}{n+1}\right)^{n^2}} = \left(\frac{n}{n+1}\right)^n = \frac{1}{\left(\frac{n+1}{n}\right)^n}$$

$$= \frac{1}{\left(1+\frac{1}{n}\right)^n} \longrightarrow \frac{1}{e} \quad \text{as} \quad n \to \infty$$

7. (12 points) Find the exact length of the curve given by

$$x(t) = 1 + 2t, \quad y(t) = e^t + e^{-t}$$

for t in the interval $[0, \ln 7]$.

$$\frac{dx}{dt} = 2 \Rightarrow \left(\frac{dx}{dt}\right)^2 = 4$$

$$\frac{dy}{dt} = e^t - e^{-t} \rightarrow \left(\frac{dy}{dt}\right)^2 = e^{2t} - 2 + e^{-2t}$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{e^{2t} + 2 + e^{-2t}} = \sqrt{\left(e^t + e^{-t}\right)^2} = e^t + e^{-t}$$

So

$$L = \int_0^{\ln 7} \left(e^t + e^{-t}\right) dt$$

$$= e^t - e^{-t} \Big|_0^{\ln 7}$$

$$= \left(e^{\ln 7} - e^{-\ln 7}\right) - \left(e^0 - e^0\right)$$

$$= 7 - \frac{1}{7}$$

$$= \frac{48}{7}$$

8. (12 points) Compute the following limit.

$$\lim_{x\to 0} \frac{\cos(x^2) + \frac{1}{2}x^4 - 1}{x^8}$$

$$\cos(x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n}}{(2n)!} = 1 - \frac{1}{2}x^4 + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \cdots$$

So $$\lim_{x\to 0} \frac{\cos(x^2) + \frac{1}{2}x^4 - 1}{x^8} = \lim_{x\to 0} \frac{-1 + \frac{1}{2}x^4 + \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{(2n)!}}{x^8}$$

$$= \lim_{x\to 0} \frac{-1 + \frac{1}{2}x^4 + \left(1 - \frac{1}{2}x^4 + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \cdots\right)}{x^8}$$

$$= \lim_{x\to 0} \frac{\frac{1}{4!}x^8 - \frac{1}{6!}x^{12} + \cdots}{x^8} = \lim_{x\to 0} \frac{1}{4!} - \frac{1}{6!}x^4 + \cdots$$

$$= \frac{1}{4!}$$

$$= \frac{1}{24}$$

9. (12 points) Let

$$f(x) = \sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2} x^n.$$

What is the radius of convergence R of this power series? What is the radius of convergence R' of the power series for $f'(x)$? For partial credit, you may write how R' is related to R.

$$\left| \frac{[2(n+1)]!}{[(n+1)!]^2} \frac{x^{n+1}}{x^n} \cdot \frac{((n)!)^2}{(2n)!} \right| = \frac{(2n+2)(2n+1)\cancel{(2n)!}}{(n+1)\cancel{(n)!}(n+1)\cancel{n!}} \cdot \frac{\cancel{n!}\,\cancel{n!}}{\cancel{(2n)!}} |x|$$

$$= \frac{2(2n+\cancel{2})(2n+1)}{(n+1)(n+1)} |x| = \frac{2(2n+1)}{n+1} |x|$$

$$= \frac{2(2+1/n)}{1+1/n} |x| \longrightarrow 4|x|$$

$$\Rightarrow \quad R = \frac{1}{4}$$

Radius of convergence of f' is same, so $R' = \frac{1}{4}$ too

10. (12 points) Find the Maclaurin series for the function

$$f(x) = x^3 \tan^{-1}(x^2).$$

What is the radius of convergence?

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$$\Rightarrow \tan^{-1}(x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{2n+1}$$

$$\Rightarrow x^3 \tan^{-1}(x^2) = x^3 \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{2n+1}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+5}}{2n+1}$$

$$\left| \frac{x^{4(n+1)+5}}{2(n+1)+1} \cdot \frac{2n+1}{x^{4n+5}} \right| = \left| \frac{x^{4n+9}}{x^{4n+5}} \right| \frac{2n+1}{2n+3} \rightarrow |x^4| < 1$$

So $\underline{\underline{R=1}}$