MA 162
Lecture 1

12.1

Basic Notation

$\mathbb{R}$ - real numbers

$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(a,b) \mid a,b \in \mathbb{R}\}$ (ordered pairs of real numbers)

$\mathbb{R}^n = \mathbb{R} \times \cdots \times \mathbb{R}$ = ordered n-tuples of real numbers

$\in$ - "in" or "member of"

$\mathbb{Z}$ - integers

A point in $\mathbb{R}^2$ is denoted by an ordered pair (a,b).
A point in $\mathbb{R}^3$ is denoted by an ordered triple (a,b,c).
We generally label points by capital letters.

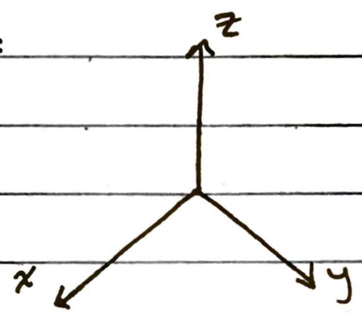
A point $P(a,b,c)$ has coordinates (a,b,c), where a is the x-coord, b is the y-coord, c is the z-coord.

In $\mathbb{R}^3$ how we draw our coordinate system matters.
It must follow the "right hand rule"

Right hand rule: two ways

1) Point your fingers in the x direction, curl them toward the y direction (into your palm), then your thumb points in the z direction.
2) Point your index finger in the x direction, your middle finger in the y direction, then your thumb points in the z direction.

Ex:



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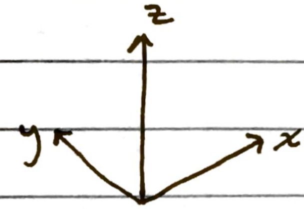

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The distance formula

Recall the distance formula in $\mathbb{R}^2$: Given points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, the distance between P_1 and P_2, denoted $|P_1P_2|$ is given by

$$|P_1P_2| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$$

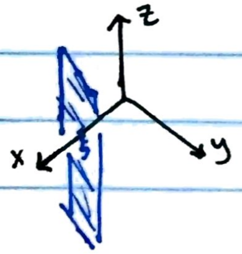
This extends naturally to any dimension. In particular, for $\mathbb{R}^3$ with $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$, the distance between P_1 and P_2 is given by

$$|P_1P_2| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2} \qquad (1)$$

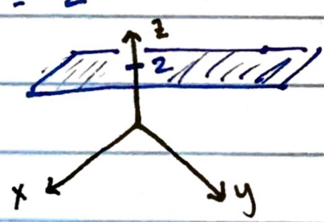
Equations in $\mathbb{R}^3$

In $\mathbb{R}^2$, if we come across an equation involving x and y, we get a curve. In $\mathbb{R}^3$, however, the same equation describes a surface.

Ex: $x = 5$

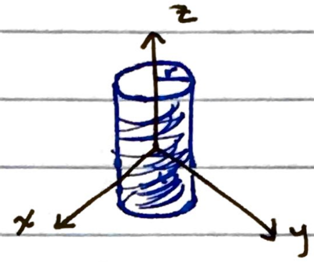


$z = 2$



Ex: $x^2 + y^2 = r^2$

Since z doesn't appear here, z is allowed to vary from $-\infty$ to ∞. In R^2, this is the equation of the circle centered at the origin with radius r. Letting z vary gives us a cylinder.



Equation of a sphere

The equation of a sphere in R^3 looks a lot like that of a circle in R^2. A sphere with center $C(h,k,l)$ and radius r is given by

$$(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2 \qquad (2)$$

or if (h,k,l) is the origin, we're left with

$$x^2 + y^2 + z^2 = r^2. \qquad (3)$$

Remark Notice that equations (2) and (3) follow immediately from the distance formula (1) after squaring both sides. That is, the sphere given by (2) is the set of (x,y,z) that are of distance r from the point (h,k,l).

Check point questions

1) Determine whether the triangle with vertices

$P(3,6,1)$, $Q(5,6,1)$, $R(3,8,1)$

is an isosceles triangle. Is it a right triangle?

2) Find ~~the~~ an equation of a sphere if one of its diameters has endpoints $(5,4,3)$ and $(1,6,-9)$.

Solutions

1) $|PQ| = \sqrt{(5-3)^2 + (6-6)^2 + (1-1)^2} = 2$

$|PR| = \sqrt{(3-3)^2 + (8-6)^2 + (1-1)^2} = 2$

$|QR| = \sqrt{(3-5)^2 + (8-6)^2 + (1-1)^2} = \sqrt{8} = 2\sqrt{2}$

isosceles? check, right triangle: $|PQ|^2 + |PR|^2 = 8 = |QR|^2$

so right triangle, too

2) The line from $(5,4,3)$ to $(1,6,-9)$ is a diameter so its midpoint is the center. The midpoint is given by

$$\left(\frac{5+1}{2}, \frac{4+6}{2}, \frac{3+(-9)}{2}\right) = (3,5,-3)$$

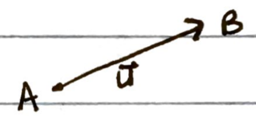
Now since $(5,4,3)$ is a point on the sphere, the distance from $(5,4,3)$ to the center $(3,5,-3)$ gives us the radius: $\sqrt{(5-3)^2 + (4-5)^2 + (3--3)^2} = \sqrt{41}$

So ~~the~~ an equation of the sphere is

$$(x-3)^2 + (y-5)^2 + (z+3)^2 = 41$$

12.2

A vector in $\mathbb{R}^n$ is an object with a magnitude (length) and direction. We generally use lower case letters like u, v to denote vectors. Sometimes with an arrow above, like $\vec{u}, \vec{v}$, or bolded when typed. We can also describe a vector by its endpoints. For example a vector $\vec{u}$ with initial point A and terminal point B can be written $\overrightarrow{AB}$

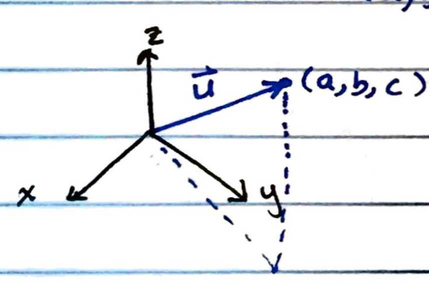


Remark By this characterization, the position of a vector is immaterial. We call two vectors with the same magnitudes and directions equivalent.

equivalent vectors

To attach a coordinate system with these vectors we require that the initial point of each vector be the origin. We call the "coordinates" of a vector its components, and we denote this by angled brackets to differentiate from points in space.

Ex: (a,b,c) $\qquad \vec{u} = \langle a,b,c \rangle$

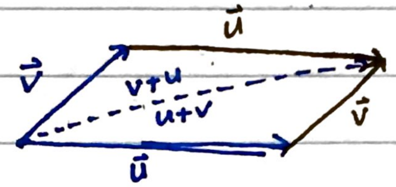


Structure of vectors

With this notion we can add vectors together. We do this the most natural way: component-wise:

$$\langle a, b, c\rangle + \langle d, e, f\rangle = \langle a+d, b+e, c+f\rangle.$$

Geometrically, vector addition follows what is called the parallelogram law.

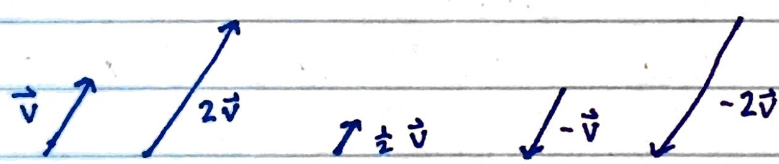


We always match head to tail.

For multiplication, we can't simply multiply vectors together, but we do have the notion of scalar multiplication. A scalar is a real number.

If c is a scalar and $\vec{v}$ is a vector, then $c\vec{v}$ is a scalar multiple of $\vec{v}$, meaning it is a vector ~~in the same direct~~ whose length is $|c|$ times that of $\vec{v}$ and whose direction is the same as $\vec{v}$ if $c>0$ and the opposite of $\vec{v}$ if $c<0$.

Ex



Remark We can find $\vec{u}-\vec{v}$ by using the parallelogram law for $\vec{u}+(-\vec{v})$

Algebraically, the vector $\vec{a}$ with representation $\overrightarrow{AB}$ is given by

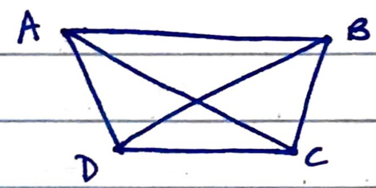
$$\langle x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1 \rangle,$$

where $A = (x_1, y_1, z_1)$ and $B = (x_2, y_2, z_2)$.

The <u>magnitude</u> (<u>length</u>) of $\vec{v}$, denoted $|\vec{v}|$ or $\|\vec{v}\|$ is given by the distance from the origin to its terminal point.

<u>Check point questions</u>

1) $\langle 3, 2, 1 \rangle + \langle -1, 7, 1 \rangle = \langle 2, 9, 2 \rangle$
2) $3\langle 1, 9, -5 \rangle = \langle 3, 27, -15 \rangle$
3) Write each combination as a single vector.
   (a) $\overrightarrow{AB} + \overrightarrow{BC}$ (b) $\overrightarrow{CD} + \overrightarrow{DB}$
   (c) $\overrightarrow{DB} - \overrightarrow{AB}$ (d) $\overrightarrow{DC} + \overrightarrow{CA} + \overrightarrow{AB}$



(a) $\overrightarrow{AC}$, (b) $\overrightarrow{CB}$, (c) $\overrightarrow{DA}$, (d) $\overrightarrow{DB}$

<u>Properties of Vectors</u>

$\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$, $a, b \in \mathbb{R}$, then

1) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
2) $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$
3) $\vec{u} + \vec{0} = \vec{u}$
4) $\vec{u} + (-\vec{u}) = \vec{0}$
5) $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$
6) $(a+b)\vec{u} = a\vec{u} + b\vec{u}$
7) $(ab)\vec{u} = a(b\vec{u})$
8) $1\vec{u} = \vec{u}$

Standard basis vectors

$\hat{i} = \langle 1, 0, 0 \rangle$, $\hat{j} = \langle 0, 1, 0 \rangle$, $\hat{k} = \langle 0, 0, 1 \rangle$

Every vector in R^3 can be represented as a combination of $\hat{i}, \hat{j}, \hat{k}$.

Ex: $\langle 3, 7, -4 \rangle = 3\hat{i} + 7\hat{j} - 4\hat{k}$.

Check point questions

1) Find a unit vector that has the same direction as the vector $\langle 6, -2 \rangle$
2) Find a vector that has the same direction as $\langle 6, 2, -3 \rangle$ but has length 4.

1) $|\langle 6, -2 \rangle| = \sqrt{36 + 4} = \sqrt{40}$

So $\frac{1}{\sqrt{40}} \langle 6, -2 \rangle$ has length 1.

2) $|\langle 6, 2, -3 \rangle| = \sqrt{36 + 4 + 9} = 7$

So $\frac{4}{7} \langle 6, 2, -3 \rangle$ has length 4.