7.4 Partial Fractions

Observe $\frac{2}{x-1} - \frac{1}{x+2} = \frac{2(x+2)-(x-1)}{(x-1)(x+2)} = \frac{x+5}{x^2+x-2}$

Then $\int \frac{x+5}{x^2+x-2}\,dx = \int \frac{2}{x-1}\,dx - \int \frac{1}{x+2}\,dx$

$$= \boxed{2\ln|x-1| - \ln|x+2| + C}$$

Idea to work this process in reverse to integrate rational functions, ie, $f(x) = \frac{P(x)}{q(x)}$.

Step 1 We need that $\deg P(x) < \deg q(x)$. If $\deg P(x) \geq \deg q(x)$ we first perform long division.

Ex 1 $\int \frac{x^3+x}{x-1}\,dx$

$$\begin{array}{r} x^2+x+2 \\ x-1 \overline{)\, x^3 \qquad +x} \\ -(x^3 - x^2) \\ \hline x^2 + x \\ -(x^2 - x) \\ \hline 2x \\ -(2x-2) \\ \hline 2 \end{array}$$

$$= \int \left[x^2+x+2 + \frac{2}{x-1}\right]dx$$

$$= \boxed{\tfrac{1}{3}x^3 + \tfrac{1}{2}x^2 + 2x + 2\ln|x-1| + C}$$

Step 2 Once we have $f(x) = \frac{P(x)}{q(x)} = S(x) + \frac{R(x)}{q(x)}$, we factorize $q(x)$ as much as possible, as a product of linear factors and irreducible quadratics.

Step 3 Write $\frac{R(x)}{q(x)}$ as sum of partial fractions of the form

$$\frac{A}{(ax+b)^i} \quad \text{and} \quad \frac{Ax+B}{(ax^2+bx+c)^j}.$$

To do this, we consider 4 cases.

Case 1 $q(x)$ is a product of linear factors.

That is, $q(x) = (a_1 x + b_1) \cdots (a_k x + b_k)$, no factor repeated and no factor a constant multiple of another. Then there exist $A_1, \ldots, A_k$ such that

$$\frac{R(x)}{q(x)} = \frac{A_1}{a_1 x + b_1} + \cdots + \frac{A_k}{a_k x + b_k}.$$

Case 2 $q(x)$ is product of linear factors, some of which are repeated. If the factor $a_1 x + b_1$ is repeated r times, then we would have

$$\frac{A_1}{a_1 x + b_1} + \frac{A_2}{(a_1 x + b_1)^2} + \cdots + \frac{A_r}{(a_1 x + b_1)^r}$$

instead of just $A_1/(a_1 x + b_1)$.

Case 3 $\frac{R(x)}{q(x)}$ contains irreducible quadratics, none of which are repeated.

Each irreducible quadratic will correspond to a term of the form

$$\frac{Ax + B}{ax^2 + bx + c}$$

in the sum of the partial fractions.

Case 4 $q(x)$ contains repeated irreducible quadratic factor. Say $(ax^2 + bx + c)^r$. Then we have

$$\frac{A_1 x + B_1}{ax^2 + bx + c} + \cdots + \frac{A_r x + B_r}{(ax^2 + bx + c)^r}$$

appears in sum of partial fractions.

Ex2 $\dfrac{x^2+2x-1}{2x^3+3x^2-2x} = \dfrac{x^2+2x-1}{x(2x-1)(x+2)}$

$= \dfrac{A}{x} + \dfrac{B}{2x-1} + \dfrac{C}{x+2}$

Ex3 $\dfrac{4x}{x^3-x^2-x+1} = \dfrac{4x}{(x-1)^2(x+1)}$

$= \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{x+1}$

Ex4 $\dfrac{1-x+2x^2-x^3}{x(x^2+1)^2} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1} + \dfrac{Dx+E}{(x^2+1)^2}$

Remark Sometimes completing the square is necessary.

Recall $\boxed{\displaystyle\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C}$

Ex5 $\displaystyle\int \frac{4x^2-3x+2}{4x^2-4x+3}\,dx$

Divide.
$$\begin{array}{r|l} & 1 \\ \hline 4x^2-4x+3 & 4x^2-3x+2 \\ & -(4x^2-4x+3) \\ \hline & x-1 \end{array}$$

$\displaystyle\int 1 + \frac{x-1}{4x^2-4x+3}$ $\qquad b^2-4ac<0 \Rightarrow$ irreducible

Complete the square: $4x^2-4x+3$

$= 4(x^2-x+\frac{3}{4})$

$= 4(x^2-x+1-\frac{1}{4})$

$= 4((x-\frac{1}{2})^2+\frac{3}{4})$

$= 4(x-\frac{1}{2})^2+2$

$= (2x-1)^2+2$

So let $u = 2x-1 \Rightarrow \frac{1}{2}(u+1) = x$
$du = 2dx$

Then $\int \frac{x-1}{4x^2-4x+3}\,dx = \int \frac{x-1}{(2x-1)^2+2} = \int \frac{\frac{1}{2}(u+1)-1}{u^2+2}\cdot\frac{1}{2}\,du$

$= \frac{1}{4}\int \frac{u-1}{u^2+2}\,du$

$= \frac{1}{4}\int \frac{u}{u^2+2}\,du - \frac{1}{4}\int \frac{1}{u^2+2}\,du$

$= \frac{1}{8}\ln(u^2+2) - \frac{1}{4}\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + C$

$= \frac{1}{8}\ln((2x-1)^2+2) - \frac{1}{4\sqrt{2}}\tan^{-1}\left(\frac{2x-1}{\sqrt{2}}\right) + C$

Remember we also had $\int 1\,dx$ + (all this)
So final answer:

$$\boxed{x + \frac{1}{8}\ln[(2x-1)^2+2] - \frac{1}{4\sqrt{2}}\tan^{-1}\left(\frac{2x-1}{\sqrt{2}}\right) + C}$$

Ex 6 $\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}\,dx$

$= \int \left(\frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}\right)dx$

Need to solve for A, B, C, D, E.

$1-x+2x^2-x^3 = A\underbrace{(x^2+1)^2}_{x^4+2x^2+1} + (Bx+C)x(x^2+1) + (Dx+E)x$
$= Ax^4 + 2Ax^2 + A + Bx^4 + Bx^2 + Cx^3 + Cx + Dx^2 + Ex$
$= (A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A$

$\Rightarrow A+B=0,\ C=-1,\ 2A+B+D=2,\ C+E=-1,\ A=1$

$\Rightarrow \boxed{B=-1}\quad \boxed{A=1}\quad \boxed{C=-1}\quad \boxed{E=0}$

$2-1+D=2 \quad \Rightarrow \boxed{D=1}$

Get $\displaystyle\int\left(\frac{1}{x}+\frac{-x-1}{x^2+1}+\frac{x}{(x^2+1)^2}\right)dx$

$\displaystyle = \ln|x| - \int\frac{x}{x^2+1}dx - \int\frac{1}{x^2+1}dx + \int\frac{x}{(x^2+1)^2}dx$

$$\boxed{= \ln|x| - \tfrac{1}{2}\ln(x^2+1) - \tan^{-1}x - \frac{1}{2(x^2+1)} + C}$$

Practice

1) $\displaystyle I=\int\frac{\sqrt{x+4}}{x}\,dx$

2) $\displaystyle\int\frac{x}{x^2(x-1)^2}\,dx$

1) $u=\sqrt{x+4} \Rightarrow u^2 = x+4$

$\Rightarrow x=-4+u^2$

$dx = 2u\,du$

$$1+\frac{4}{u^2-4}$$
$$u^2-4\,\big)\,u^2$$
$$-(u^2-4)$$
$$+4$$

So $\displaystyle I = \int\frac{u}{u^2-4}\cdot 2u\,du = 2\int\frac{u^2}{u^2-4}\,du$

$\displaystyle = 2\int\left(1+\frac{4}{u^2-4}\right)du = 2\int\left(1+\frac{4}{(u+2)(u-2)}\right)du$

$\displaystyle = 2\int\left(1+\frac{A}{u+2}+\frac{B}{u-2}\right)du$

$$4 = A(u-2) + B(u+2)$$

$u=2 \Rightarrow \quad 4 = 4B \Rightarrow B = 1$

$u=-2 \Rightarrow \quad 4 = -4A \Rightarrow A = -1$

Get

$$2\int\left(1 - \frac{1}{u+2} + \frac{1}{u-2}\right) du$$

$$= 2\left(u - \ln|u+2| + \ln|u-2|\right) + C$$

$$= 2u + 2\ln\left|\frac{u-2}{u+2}\right| + C$$

$$= \boxed{2\sqrt{x+4} + 2\ln\left|\frac{\sqrt{x+4}-2}{\sqrt{x+4}+2}\right| + C}$$

2) $I = \int \frac{1}{x^2(x-1)^2}\,dx$

$$\frac{1}{x^2(x-1)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2}$$

$$1 = Ax(x-1)^2 + B(x-1)^2 + Cx^2(x-1) + Dx^2$$

$x=0$ gives

$$1 = 0 + B + 0 + 0 \Rightarrow \boxed{B=1}$$

$x=1$ gives

$$1 = 0 + 0 + 0 + D \Rightarrow \boxed{D=1}$$

Terms corresp. to x^3: A, C

$$\Rightarrow A + C = 0 \Rightarrow A = -C.$$

Plugging in $x=2$ gives

$$1 = A(2)(1)^2 + 1(1)^2 + -A(2)^2(1) + 1(2)^2$$

$$1 = 2A + 1 - 4A + 4$$

$$2A = 4 \Rightarrow \boxed{A=2} \Rightarrow \boxed{C=-2}$$

So

$$I = \int\left(\frac{2}{x} + \frac{1}{x^2} - \frac{2}{x-1} + \frac{1}{(x-1)^2}\right)dx$$

$$= 2\ln|x| - \frac{1}{x} - 2\ln|x-1| - \frac{1}{x-1} + C$$

$$= \boxed{2\ln\left|\frac{x}{x-1}\right| - \frac{1}{x} - \frac{1}{x-1} + C}$$