Up to now we have considered integrals on finite intervals of continuous functions (or functions with only jump or removable discontinuities). We want to be able to compute integrals for infinite intervals and functions with infinite discontinuities.

Improper integrals, type 1

(a) If $\int_a^t f(x)\,dx$ exists for every $t \geq a$, then

$$\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

provided this limit exists (as a finite number).

(b) If $\int_t^b f(x)$ exists for every $t \leq b$, then

$$\int_{\infty}^{b} f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

provided this limit exists (as a finite number).

The improper integrals $\int_a^{\infty} f(x)\,dx$ and $\int_{-\infty}^{b} f(x)\,dx$ are called convergent if the corresponding limit exists and divergent if the limit does not exist.

(c) If both $\int_a^{\infty} f(x)\,dx$ and $\int_{-\infty}^{a} f(x)\,dx$ are convergent, then we define

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^{\infty} f(x)\,dx$$

In (c), any real number can be used for a.

Ex1 $\int_1^{\infty} \frac{1}{x}\,dx$

$$= \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty} \ln x\Big|_1^t = \lim_{t\to\infty} (\ln t - \ln 1)$$

$$= \lim_{t\to\infty} \ln t = \infty$$

Divergent

Ex2 $\int_1^\infty \frac{1}{x^2}\,dx$

$$= \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx = \lim_{t\to\infty} -\frac{1}{x}\Big|_1^t$$

$$= \lim_{t\to\infty} -\left(\frac{1}{t}-1\right) = \lim_{t\to\infty}\left(1-\frac{1}{t}\right) = 1.$$

So $\int_1^\infty \frac{1}{x^2}\,dx$ converges to 1. (Or we say is equal to 1)

Natural question For what values $p>0$ does the integral $\int_1^\infty \frac{1}{x^p}$ converge?

Two cases: $p=1$ and $p\neq 1$.

$p=1$ then we have $\int_1^\infty \frac{1}{x}\,dx = \infty$ by example 1.

$p\neq 1$ $\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^p}\,dx$

$$= \lim_{t\to\infty}\int_1^t x^{-p}\,dx$$

$$= \lim_{t\to\infty} \frac{1}{1-p} x^{1-p}\Big|_1^t$$

$$= \lim_{t\to\infty} \frac{1}{1-p}\left(t^{1-p}-1\right)$$

$$= \frac{1}{1-p}\left(\lim_{t\to\infty} t^{1-p}\right) - \frac{1}{1-p}$$

And $\lim_{t\to\infty} t^{1-p}$ exists if and only if $1-p<0 \Rightarrow 1<p$.

To sumarize:

$$\boxed{\int_1^p \frac{1}{x^p}\,dx \text{ converges if } p>1, \text{ diverges if } p\leq 1.}$$

Ex3 $\int_{-\infty}^\infty \frac{1}{x^2+1}\,dx$

Let's choose $a=0$ as in (c) of the definition

We have then

$$\int_{-\infty}^0 \frac{1}{x^2+1}\,dx + \int_0^\infty \frac{1}{x^2+1}\,dx$$

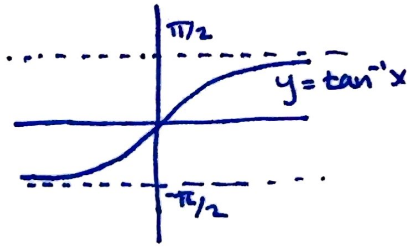
$$= \lim_{t\to-\infty}\int_t^0 \frac{1}{x^2+1} + \lim_{s\to\infty}\int_0^s \frac{1}{x^2+1}\,dx$$

$$= \lim_{t\to-\infty} \tan^{-1}x \Big|_t^0 + \lim_{s\to\infty} \tan^{-1}x \Big|_0^s$$

$$= \lim_{t\to-\infty} (\tan^{-1}0 - \tan^{-1}t) + \lim_{s\to\infty} (\tan^{-1}s - \tan^{-1}0)$$

$$= \left[0 - \left(-\tfrac{\pi}{2}\right)\right] + \left[\tfrac{\pi}{2} - 0\right]$$

$$= \boxed{\pi}$$



Type 2 Discontinuous Integrands

(a) If f is continuous on $[a,b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$$

if this limit exists. (as a finite number)

(b) If f is continuous on $(a,b]$ and is discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$$

if this limit exists (as a finite number).

The improper integral $\int_a^b f(x)$ is called convergent if the corresponding limit exists and divergent if the limit does not exist

(c) If f has a discontinuity at c, where $a<c<b$, and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, we define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Ex 4

$$\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$$

$$= \lim_{t\to 2^+} \int_t^5 (x-2)^{-1/2}\,dx = \lim_{t\to 2^+} 2(x-2)^{1/2}\Big|_t^5$$

$$= \lim_{t\to 2^+} 2\left[(5-2)^{1/2} - (t-2)^{1/2}\right]$$

$$= \boxed{2\sqrt{3}}$$

Practice

1) $\int_{-1}^{1} \frac{1}{x}$

2) $\int_{e}^{\infty} \frac{1}{x(\ln x)^2}\,dx$

3) $\int_{0}^{\pi/2} \tan^2\theta\,d\theta$

4) $\int_0^1 x\ln x\,dx$

Solutions

1) $\int_{-1}^{1} \frac{1}{x}\,dx = \int_{-1}^{0} \frac{1}{x}\,dx + \int_{0}^{1} \frac{1}{x}\,dx$

$$\int_0^1 \frac{1}{x}\,dx = \lim_{t\to 0^+} \int_t^1 \frac{1}{x}\,dx = \lim_{t\to 0^+} \ln x \Big|_t^1$$

$$= \lim_{t\to 0^+} (\ln 1 - \ln t)$$

$$= \lim_{t\to 0^+} -\ln t = \infty \Rightarrow \underline{\text{Divergent}}$$

Warning $\infty - \infty$, $0\cdot\infty$, $\frac{\infty}{\infty}$, $\frac{0}{0}$ are indeterminant forms. We can't say what they are equal to, so we need to investigate further to determine whether such a limit exists.

Another warning Had we not recognized that 1) is an improper integral, we would have gotten

$$\int_{-1}^{1} \frac{1}{x}\,dx = \ln|x| \Big|_{-1}^{1} = \ln(1) - \ln(1) = 0 - 0 = 0 \ !$$

2) Let $u = \ln x$, $du = \frac{1}{x}dx$, so

$$\lim_{t\to\infty} \int_e^t \frac{1}{x\ln(x)^2}\,dx = \lim_{t\to\infty} \int_1^{\ln t} u^{-2}\,du$$

$$= \lim_{t\to\infty} -\frac{1}{u}\Big|_1^{\ln t} = \lim_{t\to\infty} -\frac{1}{\ln t} + 1 = \boxed{1.}$$

3) $\int_0^{\frac{\pi}{2}} \tan^2\theta \, d\theta = \lim_{t\to\pi/2^-} \int_0^t \tan^2\theta \, d\theta$

$= \lim_{t\to\pi/2^-} \int_0^t (\sec^2\theta - 1)\, d\theta = \lim_{t\to\pi/2^-} (\tan\theta - \theta)\Big|_0^t$

$= \lim_{t\to\pi/2^-} \tan t - t = \infty$ DIVERGES

4) $\int_0^1 x\ln x \, dx = \lim_{t\to 0^+} \int_t^1 x \ln x \, dx$ $u = \ln x$ $dv = x\,dx$ $du = \frac{1}{x}dx$ $v = \frac{1}{2}x^2$

$= \lim_{t\to 0^+} \left[\frac{1}{2}x^2 \ln x - \int \frac{1}{2}x\,dx\right]_t^1$

$= \lim_{t\to 0^+} \left[\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2\right]_t^1$

$= \lim_{t\to 0^+} -\frac{1}{4} - \left(\frac{t^2}{2}\ln t^2 - \frac{1}{4}t^2\right)$

$= -\frac{1}{4} - \lim_{t\to 0^+} t^2 \ln t$ "$0 \cdot -\infty$" $= ?$

$= -\frac{1}{4} - \lim_{t\to 0^+} \frac{\ln t}{1/t^2}$ "$\frac{-\infty}{\infty}$" $\rightarrow$ L'Hopital

$= -\frac{1}{4} - \lim_{t\to 0^+} \frac{1/t}{-2/t^3}$

$= -\frac{1}{4} - \lim_{t\to 0^+} -\frac{1}{2}t^3 \cdot \frac{1}{t}$

$= \boxed{-\frac{1}{4}}$